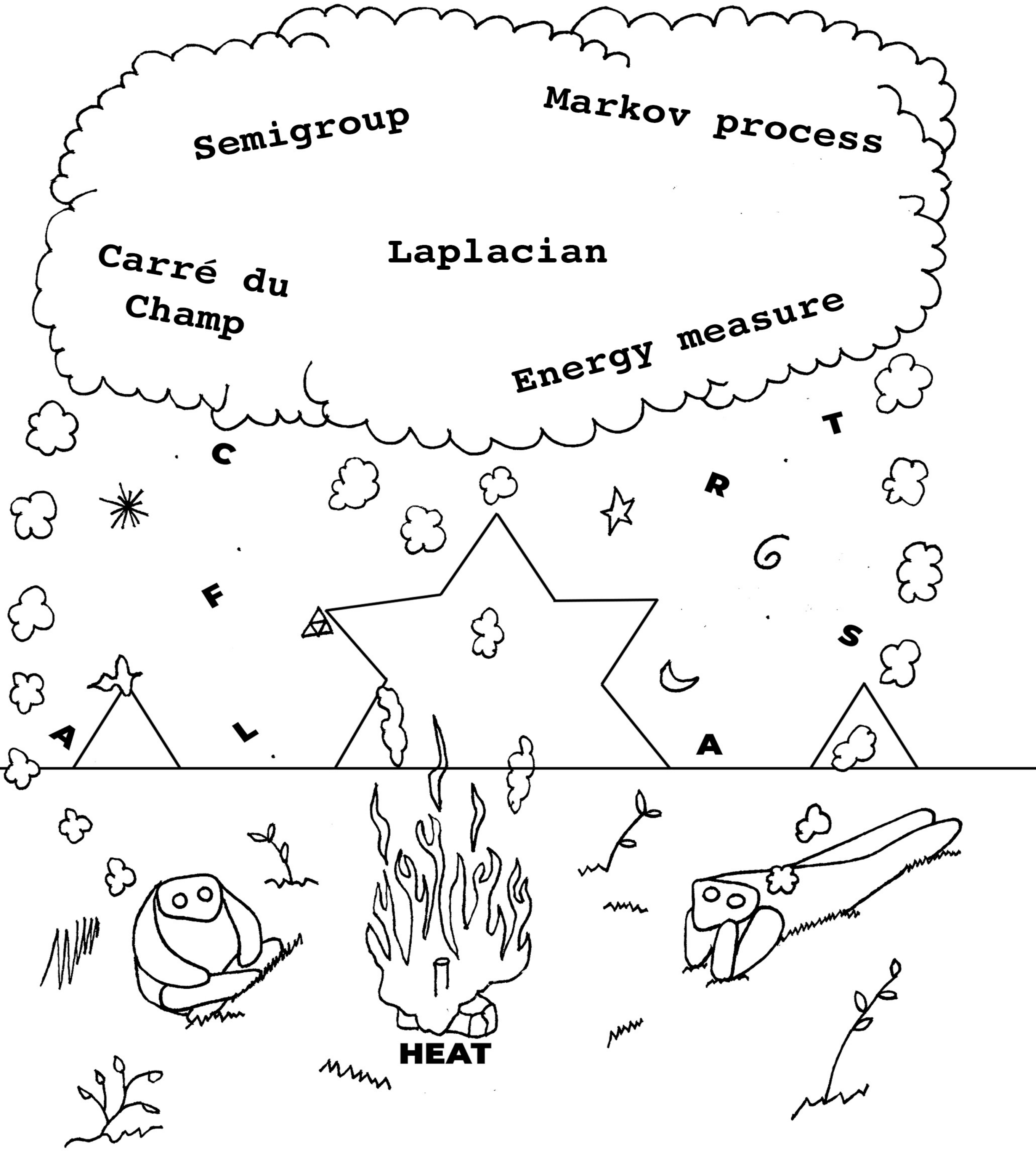
HEAT KERNEL & DIRICHLET FORMS



Heat kernels and Dirichlet Forms

Lecture notes

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CHAPTER 1

What is a heat kernel?

Working with abstract metric measure spaces or with concrete fractal spaces in my daily research has made me realize how easily one can forget the way things work in the most basic settings such as the real line. To take the audience in a journey through my research, it seems to me that precisely such a classic example may serve as bridge and lens to enter into the field with ease.

We will thus use the real line as ice-breaker in our outing to the land of heat kernels and heat semigroups, to build trust and later meet with ease non-smooth metric measure spaces like fractals ☺.

1.1. The heat equation on the real line

We owe the story of the *heat equation* (H) to the compound work of many protagonists, including Robert Brown (botanist), Albert Einstein (Physicist) and Norbert Wiener (mathematician). The quantity $u(t,x)$ whose dynamics are modeled by the partial differential equation (H) represents the temperature at a certain time $t \geq 0$ and point within the object under investigation, in our case $x \in \mathbb{R}$.

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) & x \in \mathbb{R} \\ u(0,x) = u_0(x), \end{cases} \tag{H}$$

where u_0 represents the initial temperature of the infinite long cord $(-\infty, +\infty)$.

One of the most common ways to solve (H) relies on Fourier analysis. Our convention for the Fourier transform of a function $u \in L^1(\mathbb{R}, dx)$ will be

$$\mathcal{F}(v(x))(\xi) := \widehat{v}(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x} v(x)\, dx. \tag{1.1.1}$$

We refer to [23, Chapter 9] for its many properties, in particular

$$\widehat{\frac{d}{dx}v}(\xi) = -i\xi\widehat{v}(\xi).$$

Taking the Fourier transform everywhere in (H) we arrive at the equation

$$\begin{cases} \frac{d}{dt}\widehat{u}(t,\xi) = -\frac{1}{4}\xi^2\widehat{u}(t,\xi) \\ \widehat{u}(0,\xi) = \widehat{u_0}(\xi). \end{cases} \tag{1.1.2}$$

Looking closely at the right hand side of this equation we realize the "magic" of the Fourier transform: the space-derivative disappeared! Now we are left with an ordinary differential equation, which can be solve through the Ansatz

$$\widehat{u}(t,\xi) = e^{-\frac{1}{4}\xi^2 t}\widehat{u_0}(\xi) \tag{1.1.3}$$

and the solution to the original equation (H) will materialize by taking the inverse Fourier transform of (1.1.3).

Exercise 1.1.1. Show that the inverse Fourier transform of (1.1.3) equals

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\widehat{u}(t,\xi)\,d\xi = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}u_0(y)\,dy.$$

For the sake of completeness, and to see how/where the dimension of the underlying space plays a role, similar computations to those in Exercise 1.1.1 yield

$$u(t,x) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}}e^{-\frac{\|x-y\|^2}{4t}}u_0(y)\,dy \tag{1.1.4}$$

as solution to the heat equation in $\mathbb{R}^n$. Note that the partial second derivative ∂_x^2 will be replaced by its multi-dimensional version, the Laplacian $\Delta u = \sum_{i=1}^n \partial_{x_i}^2 u$.

Definition 1.1.2. The family of functions $\{p_t\}_{t\geq 0}$ given by

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{\|x-y\|^2}{4t}}, \qquad x,y\in\mathbb{R}^n, \tag{1.1.5}$$

is called the standard (probabilistic, Gaussian) heat kernel in $\mathbb{R}^n$.

Intuitively, although this is not mathematically rigorous, one may think of $p_t(x,y)$ as the probability that heat moves from x to y at time t.

Now, what are the main properties of these functions $p_t(x,y)$?

Proposition 1.1.3. *For any $t\geq 0$, $x,y\in\mathbb{R}^n$ and $u_0\in L^2(\mathbb{R}^n,dx)$,*

(i) $p_t(x,y)\geq 0$ *(non-negative),*

(ii) $p_t(x,y) = p_t(y,x)$ *(symmetric),*

(iii) $\int_{\mathbb{R}^n} p_t(x,y)\,dy = 1$ *(stochastic completeness),*

(iv) $p_{s+t}(x,y) = \int_{\mathbb{R}^n} p_s(x,z)p_t(z,y)\,dz$ *(semigroup property),*

(v) $\int_{\mathbb{R}^n} p_t(\cdot,y)u_0(y)\,dy \xrightarrow[t\to 0^+]{L^2} u_0$.

Proof. Items (i) and (ii) follow from direct inspection. The square of the integral in (iii) can be computed directly using polar coordinates and equals one. The semigroup property follows from the identity

$$\frac{1}{(4\pi(t_1+t_2))^{n/2}}e^{-\frac{\|x-y\|^2}{4(t_1+t_2)}} = \frac{1}{(4\pi t_1)^{n/2}}\frac{1}{(4\pi t_2)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{\|x-z\|^2}{4t_1}}e^{-\frac{\|y-z\|^2}{4t_2}}dz.$$

The limit in the last item is proved first pointwise for bounded continuous functions, and afterwards extended by density to L^2-functions. We show here the pointwise limit

$$\lim_{t\to 0^+}\int_{\mathbb{R}_n} p_t(x,y)u_0(y)\,dy = u_0(x).$$

Applying (iii), polar coordinates and a change of variables, for any $M>0$

$$\begin{aligned}
&\Big|u(x)-\int_{\mathbb{R}_n} p_t(x,y)u_0(y)\,dy\Big|\\
&=\Big|\frac{1}{(4\pi t)^{n/2}}\int_0^\infty e^{-\frac{r^2}{4t}}r^{d-1}\int_{\mathcal{S}^{n-1}}\big(u(x)-u(x+r\theta)\big)\,d\sigma(\theta)\,dr\Big|\\
&\le\Big|\frac{1}{\pi^{n/2}}\int_0^M e^{-s^2}s^{d-1}\int_{\mathcal{S}^{n-1}}\big(u(x)-u(x+2\sqrt{t}s\theta)\big)\,d\sigma(\theta)\,ds\Big|\\
&+\Big|\frac{1}{\pi^{n/2}}\int_M^\infty e^{-s^2}s^{d-1}\int_{\mathcal{S}^{n-1}}\big(u(x)-u(x+2\sqrt{t}s\theta)\big)\,d\sigma(\theta)\,ds\Big|\\
&\le\sup_{|y-x|\le 2\sqrt{t}M}|u(x)-u(y)|+\|u\|_{L^\infty}\frac{2\mathrm{vol}(\mathbb{S}^{n-1})}{\pi^{n/2}}\int_M^\infty e^{-s^2}s^{n-1}ds.
\end{aligned}$$

For any $\varepsilon>0$ we may choose M large enough to bound the second term by $\varepsilon/2$, and afterwards choose $t_0>0$ so that the first term is also bounded by $\varepsilon/2$ for all $0<t<t_0$. □

1.2. Heat kernels

The properties of the standard Gaussian heat kernel on $\mathbb{R}^n$ are going to serve as guidelines to define the concept of heat kernel in the more general setting of a metric measure space (M,d,μ). For the moment we only ask the space to be locally compact and separable, with the measure μ Radon (i.e. finite on compact sets, outer regular on Borel sets, and inner regular on open sets) and of full support. The following definition can be found in many sources, we refer here to [10].

Definition 1.2.1. A heat kernel $\{p_t\}_{t\ge 0}$ is a family of $\mu\otimes\mu$-measurable functions such that, for each $t\ge 0$

(i) $p_t(x,y)\ge 0$ for μ-a.e. $x,y\in M$,

(ii) $p_t(x,y)=p_t(y,x)$ for μ-a.e. $x,y\in M$,

(iii) $\int_M p_t(x,y)\,d\mu(y)=1$ for μ-a.e. $x\in M$,

(iv) $p_{s+t(x,y)}=\int_M p_s(x,z)p_t(y,z)\,d\mu(z)$ for any $0\le s\le t<\infty$ and μ-a.e. $x,y\in M$,

(v) $\displaystyle\int_M p_t(\cdot,y)u_0(y)\,d\mu(y) \xrightarrow[t\to 0^+]{L^2} u_0$.

For some it may be noteworthy to point out that, in applications, heat kernels rarely have an explicit formula like (1.1.5). Most of the times it is possible to prove their existence and approximate behavior in terms of being bounded above and below by some explicit function.

1.3. Heat semigroups

We continue under the guidance of the real line. Having seen the expression (1.1.4) of the solution to the heat equation in terms of the Gaussian heat kernel, we will start by investigating some properties of the integral operators induced by a heat kernel $p_t(x,y)$.

Proposition 1.3.1. *Let $\{p_t\}_{t\geq 0}$ denote a heat kernel and define*

$$\begin{aligned} P_t\colon L^2(M,\mu) &\longrightarrow L^2(M,\mu)\\ u &\longmapsto \int_M p_t(\cdot,y)u(y)\,d\mu(y). \end{aligned} \tag{1.3.1}$$

Then, for any $u,v\in L^2(M,\mu)$ and $t\geq 0$

(i) $P_tu\geq 0$ if $u\geq 0$,

(ii) $\langle u,P_tv\rangle_{L^2}=\langle P_tu,v\rangle_{L^2}$,

(iii) $\|P_tu\|_{L^2}\leq\|u\|_{L^2}$,

(iv) $P_{t+s}u=P_tP_su$ for any $s,t\geq 0$,

(v) $\|P_tu-u\|_{L^2}\xrightarrow[t\to 0^+]{} 0$.

Proof. Properties (i) and (ii) are direct consequences of the definition (1.3.1). Property (iii) follows from the symmetry of $p_t(x,y)$ and (iv) from Cauchy-Schwarz and Definition 1.2.1 (iii). Property (iv) is Definition 1.2.1 (v). □

Let us now introduce the formal definition of a (heat) semigroup, again emphasizing that we take as underlying function space $L^2(M,\mu)$ for the purposes of these notes, although the general theory of semigroups works in setting of Banach spaces.

Definition 1.3.2. A semigroup is a family of bounded linear operators $\{P_t\}_{t\geq 0}$ in $L^2(M,\mu)$ that satisfy

$$P_0=\mathrm{Id} \qquad \text{and} \qquad P_{t+s}=P_tP_s. \tag{1.3.2}$$

Looking at this definition we recognize that the integral operators induced by heat kernels as in (1.3.1) are indeed semigroups ☺.

Definition 1.3.3. A semigroup $\{P_t\}_{t\geq 0}$ is called

(i) *strongly continuous* if

$$\|P_tu-u\|_{L^2}\xrightarrow[t\to 0^+]{} 0 \qquad \Leftrightarrow \qquad u\in L^2(M,\mu). \tag{1.3.3}$$

(ii) *contractive* if

$$\|P_t u\|_{L^2} \leq \|u\|_{L^2} \qquad \forall\, u \in L^2(M,\mu), t \geq 0. \tag{1.3.4}$$

Strong continuity is a key property when it comes to link a semigroup with an operator that will naturally act as the second derivative we saw in the heat equation (H).

Lemma 1.3.4. *Let $\{P_t\}_{t\geq 0}$ be a strongly continuous contraction semigroup in $L^2(M,\mu)$. Then,*

(i) $P_t u \in L^2(M,\mu)$ for all $u \in L^2(M,\mu)$.

(ii) $\|P_{t+h}u - P_t u\|_{L^2} \xrightarrow[h\to 0^+]{} 0$ for all $u \in L^2(M,\mu)$ and $t \geq 0$.

Proof. (i) follows directly from contractivity (1.3.4). To prove (ii), applying the semigroup property (1.3.2), (i), contractivity and strong continuity

$$\|P_h(P_t u) - P_t u\|_{L^2} = \|P_t(P_h u - u)\|_{L^2} \leq \|P_h u - u\|_{t\to\infty} \xrightarrow[h\to 0^+]{} 0.$$

□

1.4. Infinitesimal generators

As mentioned in the previous section, the infinitesimal generator of a strongly continuous contractive semigroup is the operator that will naturally act as the second derivative in the heat equation. Before going into precise definitions, let us take a short tour over differentiation and integration properties of L^2-valued functions. For a strongly continuous contractive semigroup $\{P_t\}_{t\geq 0}$ on $L^2(M,\mu)$ we define

∘ *Differentiation:* For any $u \in L^2(M,\mu)$,

$$\frac{d}{ds}P_s u := \lim_{h\to 0}\frac{1}{h}(P_{s+h}u - P_s u).$$

∘ *Integration:* For any interval $[a,b]$ with $0 \leq a < b < \infty$, consider for each $n \geq 1$ a partition given by $a = t_0 < t_1 < \ldots < t_n = b$, so that $\max\limits_{1\leq k<n} |t_k - t_{k-1}| \xrightarrow{n\to\infty} 0$.

For any $u \in L^2(M,\mu)$, define

$$\int_b^a P_s u ds := s\text{--}\lim_{n\to\infty}\sum_{k=1}^n (t_k - t_{k-1})P_{s_k}u,$$

where $s_k \in [t_k, t_{k-1}]$ for every $1 \leq k \leq n$.

The next lemma collects useful properties of differentiation and integration for strongly continuous semigroups that we will use later on.

Lemma 1.4.1. *Let $\{P_t\}_{t\leq 0}$ be a strongly continuous semigroup on $L^2(M,\mu)$, then*

(i) *For all $f \in L^2(M,\mu)$ and $s \geq 0$,*

$$\left\|\int_a^b P_s f ds\right\|_{L^2} \leq \int_a^b \|P_s f\|_{L^2}\, ds.$$

(ii) *For all* $f \in L^2(M,\mu)$ *and* $t \geq 0$,

$$\text{s-}\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} P_s f\,ds = P_t f.$$

(iii) *For all* $f \in L^2(M,\mu)$, $0 \leq a < b < \infty$ *and* $t \geq 0$,

$$P_t\left(\int_a^b P_s f\,ds\right) = \int_a^b P_{t+s} f\,ds = \int_{a+t}^{b+t} P_s f\,ds.$$

(iv) *For all* $f \in L^2(M,\mu)$, $0 \leq a < b < \infty$ *and* $s \geq 0$,

$$\int_a^b \tfrac{d}{ds} P_s f\,ds = P_b f - P_a f.$$

All the properties stated above hold replacing $L^2(M,\mu)$ by an abstract Banach space.

Definition 1.4.2. The *infinitesimal generator* of a strongly continuous semigroup $\{P_t\}_{t\geq 0}$ on $L^2(M,\mu)$ is the closed linear operator

$$Lu := \text{s-}\lim_{t\to 0^+}\frac{1}{t}(P_t u - u) \tag{1.4.1}$$

with domain $D(L) = \{u \in L^2(M,\mu)\colon$ the limit in (1.4.1) exists$\}$.

It will become apparent in Section 1.6 why one often refers to infinitesimal generator L as the *Laplacian*. Let us finish this section by checking some of its main properties.

Proposition 1.4.3. *Let* $\{P_t\}_{t\geq 0}$ *be a strongly continuous semigroup on* $L^2(M,\mu)$ *with infinitesimal generator* $(L, D(L))$.

(i) $\overline{D(L)} = L^2(M,\mu)$,

(ii) *For any* $u \in D(L)$, $Lu \in L^2(M,\mu)$.

(iii) *For any* $u \in L^2(M,\mu)$, $\int_0^t P_s u\,ds \in D(L)$ *and* $P_t u - u = L\Big(\int_0^t P_s u\,ds\Big)$.

Proof. For (i), see e.e [21, Proposition 1.10]. To prove (ii), if $u \in D(L)$, then $u \in L^2(M,\mu)$ and thus Lemma 1.3.4(ii) implies $\frac{1}{h}(P_h u - u) \in L^2(M,\mu)$ for any $h > 0$. Since the operator L is closed, the limit $Lu = \text{s-}\lim_{h\to 0^+}\frac{1}{h}(P_h u - u)$ exists and is in $L^2(M,\mu)$. □

1.5. Hille-Yosida

The words "Hille-Yosida" encompass results that put strongly continuous contraction semigroups, operators and *resolvents*, which are defined as

$$R_\lambda = (\lambda I - L)^{-1}, \qquad \lambda \in \rho(L),$$

where L is the associated (closed) operator and $\rho(L)$ is the *resolvent set* of L given by

$$\rho(L) := \{\lambda \in \mathbb{R}\colon\ (\lambda I - L)^{-1}\text{exists as bounded linear operator}\}.$$

We will not enter into further details regarding these operators and refer the interested reader to [21, Chapter 1].

The version of the Hille-Yosida theorem that we present here is tailored to the Hilbert space $L^2(M,\mu)$, although please keep in mind that it holds more generally for closed operators in a Banach space.

Theorem 1.5.1. *A closed operator $(L, D(L))$ with $D(L) \subset L^2(X,\mu)$ is the infinitesimal generator of a strongly continuous contraction semigroup $\{P_t\}_{t\geq 0}$ on $L^2(X,\mu)$ if and only if*

(i) *$D(L)$ is dense in $L^2(X,\mu)$ with respect to the L^2-norm,*

(ii) *$(0,\infty) \subset \rho(L)$ and its resolvent satisfies*

$$\|\lambda R_\lambda\|_{L^2} \leq 1 \qquad \forall\, \lambda > 0.$$

Proof. See [21, Theorem 1.12]. □

How does the theorem actually relate an operator $(L, D(L))$ and the semigroup of which it is the infinitesimal generator? The connection is hidden in the proof ☺. There, one ends up proving that

$$P_t = e^{tL} = \sum_{k=0}^{\infty} \frac{t^k}{k!} L^k$$

for any $t \geq 0$. The series above is also understood as the strong limit $\text{s}-\lim\limits_{n\to\infty} \sum_{k=0}^{n} \frac{t^k}{k!} L^k$.

Remark 1.5.2. The spectral theorem, see e.g. [5, Theorem 2.5.2] allows one to understand powers of an operator (including fractional powers) like L^k.

1.6. Back to the heat equation

We started this chapter with the heat equation on the line

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x), & x \in \mathbb{R} \\ u(0,x) = u_0(x). \end{cases} \tag{H}$$

Its solution, whose expression we found in (1.1.4), could now abstractly be written as

$$u(t,x) = \int_{\mathbb{R}} p_t(x,y) u_0(y)\, dy = P_t u_0(x), \tag{1.6.1}$$

where $p_t(x,y)$ is the standard Gaussian heat kernel (1.1.5) and P_t the associated semigroup.

Exercise 1.6.1. Show that the infinitesimal generator of the semigroup (1.6.1) coincides with the operator $u(x) \mapsto \frac{1}{2}u''(x)$, i.e.

$$Lu = \text{s}-\lim_{t\to 0^+} \frac{1}{t}(P_t u - u) = \frac{1}{2}u''$$

and $D(L)$ coincides with the Sobolev space

$$H^2(\mathbb{R}) := \{u \in L^2(\mathbb{R}, dx)\colon u', u'' \in L^2(\mathbb{R}, dx)\}.$$

The latter exercise tells us that we may rewrite (H) abstractly as

$$\begin{cases} \partial_t u(t,x) = Lu(t,x), & x \in \mathbb{R} \\ u(0,x) = u_0(x), \end{cases}$$

which together with (1.6.1) says

$$\frac{d}{dt} P_t u_0 = L P_t u_0$$

as long as $P_t u_0 \in D(L)$.

This is pretty nice! In fact we will see in this section that the equality above also holds when the underlying space is a generic metric measure space (M, d, μ) equipped with a strongly continuous contraction semigroup ☺.

Theorem 1.6.2. *Let $\{P_t\}_{t\geq 0}$ be a strongly continuous contractive semigroup in $L^2(M,\mu)$ with infinitesimal generator $(L, D(L))$. Then, for any $u \in D(L)$, we have $P_t u \in D(L)$ and*

$$\frac{d}{dt} P_t u = L P_t u = P_t L u. \tag{1.6.2}$$

In particular,

$$P_t u - u = \int_0^t L P_s u \, ds = \int_0^t P_s L u \, ds \tag{1.6.3}$$

for all $u \in D(L)$.

Proof. We start with the second equality in (1.6.2). By definition of L, for any $u \in D(L)$

$$\begin{aligned} L P_t u &= \text{s-}\lim_{h\to 0^+} \frac{1}{h}(P_{t+h} u - P_t u) = \text{s-}\lim_{h\to 0^+} P_t\Big(\frac{1}{h}(P_h u - u)\Big) \\ &= P_t\Big(\text{s-}\lim_{h\to 0^+} \frac{1}{h}(P_h u - u)\Big) = P_t L u. \end{aligned}$$

To prove the first equality in (1.6.2), notice that while

$$\text{s-}\lim_{h\to 0^+} \frac{1}{h}(P_h P_t u - P_t u) = L P_t u$$

holds by definition, it still remains to show that the same is true for the left-limit. By replacing h by $-h$ above,

$$\text{s-}\lim_{h\to 0^-} \frac{1}{h}(P_{t+h} u - P_t u) = s\text{-}\lim_{h\to 0^+} \frac{1}{h}(P_t u - P_{t-h} u).$$

How does that equal $P_t L u$? Due to the contraction property of the semigroup, the definition of L and the strong continuity of the semigroup

$$\begin{aligned} \Big\|\frac{1}{h}(P_t u - P_{t-h} u) - L P_t u\Big\|_{L^2} &= \Big\|\frac{1}{h}(P_{t-h} P_h u - P_{t-h} u) - P_t L u\Big\|_{L^2} \\ &\leq \big\|P_{t-h}\big\| \Big\|\frac{1}{h}(P_h u - u) - P_h L u\Big\|_{L^2} \\ &\leq \Big\|\frac{1}{h}(P_h u - u) - L u\Big\| + \big\|L u - P_h L u\big\|_{L^2} \xrightarrow{h\to 0^+} 0. \end{aligned}$$

By Proposition 1.4.3 (iii), the definition of L and Lemma 1.3.4 (i) and (ii) we have

$$\begin{aligned}
&\Big\|P_t u - u - \int_0^t LP_s u\,ds\Big\|_{L^2}\\
&\leq \Big\|P_t u - u - \frac{1}{h}(P_h - I)\int_0^t P_s u\,ds\Big\|_{L^2}\\
&+ \Big\|\frac{1}{h}(P_h - I)\int_0^t P_s u\,ds - \int_0^t LP_s u\,ds\Big\|_{L^2}\\
&\leq \Big\|\int_0^t P_s(\frac{1}{h}(P_h - I)u - Lu)ds\Big\|_{L^2} + O(h)\\
&\leq \int_0^t \Big\|\frac{1}{h}(P_h - I)f - Lf\Big\|ds + O(h) \xrightarrow{h\to 0^+} 0,
\end{aligned}$$

which proves the first equality in (1.6.3). The second equality follows by the second equality in (1.6.2). □

CHAPTER 2

What is a Dirichlet form?

2.1. The heat equation and symmetric forms

Looking back at Theorem 1.6.2, we have found that $P_t u_0$ solves the heat equation with initial value u_0 as long as $u_0 \in D(L)$. What is this domain in our guiding example $\mathbb{R}$? Here is when the *weak formulation* of the heat equation comes into play.

To formulate (H) weakly, we take a family of test functions $\{\varphi\}$ which is dense in $L^2(\mathbb{R}, dx)$ and "test" each side of the equality in the equation. Smooth functions vanishing at infinity, $C_0^\infty(\mathbb{R})$, are a common useful choice. In this way, for any $\varphi \in C_0^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \partial_t u \, \varphi \, dx = \frac{1}{2} \int_{\mathbb{R}} u'' \varphi \, dx. \tag{2.1.1}$$

Here we see that a solution u to the heat equation belongs to the Sobolev space $H^2(\mathbb{R})$. Working with PDEs, that is usually a lot of regularity to ask. How does one go about posing the same problem allowing less regular solutions? Applying integration by parts to the right hand side of the equation above,

$$\int_{\mathbb{R}} \partial_t u \, \varphi \, dx = -\frac{1}{2} \int_{\mathbb{R}} u' \cdot \varphi' \, dx.$$

In this formulation, u only needs one derivative in $L^2(\mathbb{R}, dx)$, which means $u \in H^1(\mathbb{R})$. Great news for lowering regularity! In fact, the functional

$$\mathcal{E}(u, v) := \frac{1}{2} \int_{\mathbb{R}} u' \cdot v' \, dx, \qquad u, v \in H^1(\mathbb{R})$$

happens to define a Dirichlet form ☺.

Its precise definition when the underlying space is our locally compact separable metric measure space (M, d, μ) will appear in Section 2.4. Before getting there, let us back up one step and talk first about symmetric forms in $L^2(M, \mu)$.

Definition 2.1.1. A symmetric form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(M, \mu)$ is a densely defined, symmetric, non-negative bilinear form, that is

(i) $D(\mathcal{E})$ is dense in $L^2(M,\mu)$ with respect to the L^2-norm.

(ii) $\mathcal{E}(u,v)=\mathcal{E}(v,u)$ for any $u,v\in D(\mathcal{E})$.

(iii) $\mathcal{E}(u,u)\geq 0$ for all $u\in D(\mathcal{E})$.

(iv) $\mathcal{E}(au+bv,w)=a\mathcal{E}(u,v)+\mathcal{E}(v,w)$ for any $u,v,w\in D(\mathcal{E})$ and $a,b\in\mathbb{R}$.

Such a form is called *closed* if

$$\|u\|_{\mathcal{E}_1}:=\big(\mathcal{E}(u,u)+\|u\|^2_{L^2}\big)^{1/2}$$

defines a norm in $D(\mathcal{E})$ and $(D(\mathcal{E}),\|\cdot\|_{\mathcal{E}_1})$ is Hilbert.

Theorem 2.1.2. *Let $(\mathcal{E},D(\mathcal{E}))$ be a closed form on $L^2(M,\mu)$, There exists a unique non-positive densely defined self-adjoint operator L such that*

$$D(L)=\{u\in D(\mathcal{E})\colon\ \exists\, h\in L^2(M,\mu)\ s.t.\ \mathcal{E}(u,v)=\langle h,v\rangle\ \ \forall\, v\in L^2(M,\mu)\}.$$

In particular, for any $u\in D(L)$,

$$\mathcal{E}(u,v)=\langle -Lu,v\rangle:=\int_M -Lu\cdot v\,d\mu\qquad\forall\, v\in D(\mathcal{E}).$$

Proof. The main idea consists in considering the bilinear forms

$$\mathcal{E}_\lambda(u,u)=\mathcal{E}(u,u)+\lambda\langle u,v\rangle$$

and apply the Riesz representation theorem to construct a family of resolvents $\{R_\lambda\}_{\lambda>0}$ whose associated generator will be L. For more details see e.g. [8, Theorem 1.3.1(ii)]. □

2.2. Symmetric forms and semigroups

We have just learned from Theorem 2.1.2 that a closed form in $L^2(M,\mu)$ "produces" a non-positive definite self-adjoint operator $(L,D(L))$. In fact, by being self-adjoint, the operator is also closed, see e.g. [5, Section 1.2]. Thinking back for a moment about the previous chapter and the Hille-Yosida theorem...

... we realize that we actually know how to obtain a strongly continuous contraction semigroup associated with $(L,D(L))$! Namely,

$$P_t u=e^{tL}u\qquad u\in L^2(M,\mu)$$

☺. How about relating directly a semigroup $\{P_t\}_{t\geq 0}$ with a symmetric form $(\mathcal{E},D(\mathcal{E}))$?

Theorem 2.2.1. *Let $(\mathcal{E},D(\mathcal{E}))$ be a closed form on $L^2(M,\mu)$ with associated generator $(L,D(L))$. Then, there exists a strongly continuous contraction semigroup $\{P_t\}_{t\geq 0}$ in $L^2(M,\mu)$ with $(L,D(L))$ as infinitesimal generator that satisfies*

$$\begin{aligned}&D(\mathcal{E})=\{u\in L^2(M,\mu)\colon\ \ \mathcal{E}(u,u)<\infty\}\\&\mathcal{E}(u,v)=\lim_{t\to 0}\frac{1}{t}\langle u-P_tu,v\rangle.\end{aligned}\tag{2.2.1}$$

Proof. The main idea is to apply Hille-Yosida to the resolvent of L and use the spectral theorem to prove the existence of the limit in (2.2.1). See e.g. [8, Lemma 1.3.4]. □

Thus far we have introduced what we are generically calling a "symmetric form" and you might be wondering: Where are the Dirichlet forms? That question bring us to a crucial word in this chapter: (being) *Markov*.

2.3. Markov processes and Markov semigroups

A deep result in the theory of Dirichlet forms is their connection to Markov processes and hence to *Markov* semigroups. In this section we will review some basic termnology from the theory of stochastic processes and explain how they link to semigroups.

To enter in the probabilistic setting we need a measurable space $(\Omega, \mathcal{F})$ consisting of a set Ω and a σ-algebra $\mathcal{F}$ of Ω. This space contains "anything that may happen whose likelihood could be measured". A random variable is just a function $X\colon \Omega \to M$ that is $\mathcal{F}$-measurable.

Equipped with a *probability measure* $\mathbb{P}$, the distribution of a random variable $X\colon \Omega \to M$ is the measure given by $\mathbb{P}(X^{-1}(B)) =: \mathbb{P}\circ X^{-1}(B)$ for any measurable set $B \subseteq M$. By a change of variables, the expectation of any function $u\colon M \to \mathbb{R}$ of a random variable $X\colon \Omega \to M$ corresponds to the integral

$$\mathbb{E}[u(X)] := \int_\Omega u(X(\omega))\, d\mathbb{P}(\omega) = \int_M u(x)\, d(\mathbb{P}\circ X^{-1})(x). \tag{2.3.1}$$

A *stochastic process* in (M, d, μ) is a family of random variables $\{X_t\}_{t\geq 0}$. For each fixed $\omega \in \Omega$, the map $t \mapsto X_t(\omega)$ is called a *path* of X_t corresponding to ω. Fixing ω would be like fixing a specific "universe": randomness disappears and we are left with a (deterministic) function of time. Note that the probability measure $\mathbb{P}$ plays no role here!

In the context of stochastic processes, the "overarching σ-algebra" $\mathcal{F}$ is usually equipped with a *filtration*, that is a family of sub-σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ with the property that

$$\mathcal{F}_t \subset \mathcal{F} \quad \forall\, t \geq 0 \qquad \text{and} \qquad \mathcal{F}_s \subseteq \mathcal{F}_t \quad \forall\, 0 \leq s \leq t. \tag{2.3.2}$$

A stochastic process $\{X_t\}_{t\geq 0}$ is *adapted* to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ when X_t is $\mathcal{F}_t$-measurable for any $t \geq 0$.

The last concept to introduce Markov processes is that of *conditional expectation* of a random variable like X_t with respect to a (sub)-σ-algebra like $\mathcal{F}_t$. That expectation is written $\mathbb{E}[X_t \,|\, \mathcal{F}_t]$. Despite the name and the notation, unlike (2.3.1), the conditional expectation *is a random variable*. From an analytic point of view, this is the Radon-Nykodym derivative of the measure $\mathbb{P}\circ X^{-1}$ and in particular

$$\int_A X(\omega)\, d\mathbb{P}(\omega) = \int_A \mathbb{E}[X_t \,|\, \mathcal{F}_t](\omega)\, d\mathbb{P}(\omega)$$

for any $A \in \mathcal{F}_t$.

All the previous concepts come together when defining a Markov process.

Definition 2.3.1. Let $(\Omega, \mathcal{F})$ be a measurable space, $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration of $\mathcal{F}$, and $\{\mathbb{P}_x\}_{x\in M}$ a family of probability measures on $(\Omega, \mathcal{F})$. A Markov process (on (M, d, μ)) is a family of (M-valued) random variables $\{X_t\}_{t\geq 0}$ that satisfy

(i) $\{X_t\}_{t\geq 0}$ is $\mathcal{F}_t$-adapted, that is, for any $t \geq 0$, X_t is $\mathcal{F}_t$-measurable.

(ii) For any bounded measurable function $u\colon M \to \mathbb{R}$, $0 \leq s \leq t$ and $x \in M$ it holds that

$$\mathbb{E}_x[u(X_t) \,|\, \mathcal{F}_s] = \mathbb{E}_x(u(X_t) \,|\, \sigma(X_s)], \tag{2.3.3}$$

where $\sigma(X_s)$ is the smallest σ-algebra with respect to which X_s is measurable, and $\mathbb{E}_x[\cdot]$ denotes the expectation with respect to the probability measure $\mathbb{P}_x$.

Observation 2.3.2. A stochastic process $\{X_t\}_{t\geq 0}$ is always adapted to the filtration given by

$$\mathcal{F}_t := \sigma\big(X_\tau\colon \tau \leq t\} \qquad t \geq 0,$$

where $\mathcal{F}_t$ is the smallest σ-algebra with respect to which all random variables $\{X_\tau\}_{\tau\leq t}$ are measurable. The family $\{\mathcal{F}_t\}_{t\geq 0}$ builds what is called the natural *filtration* for the process $\{X_t\}_{t\geq 0}$ taking as "overarching σ-algebra"

$$\mathcal{F} := \bigcup_{t\geq 0} \mathcal{F}_t.$$

Definition 2.3.3. The transition kernel associated with a Markov process $\{X_t\}_{t\geq 0}$ is the measure given by

$$p(t, x, A) := \mathbb{P}_x(X_t \in A) \qquad t \geq 0, x \in M, A \in \mathcal{B}(M).$$

In particular, $p(0, x, M) = 1$ for any $x \in M$. In the context of general transition kernels, the latter is referred to as the *Markov* property.

Theorem 2.3.4. *Given a Markov process $\{X_t\}_{t\geq 0}$ with transition kernel $p(t, x, \cdot)$, the family of operators defined as*

$$P_t u(x) := \mathbb{E}_x[u(X_t)] = \int_M u(y)p(t, x, dy) \tag{2.3.4}$$

for any $u \in L^2(M, \mu)$, is a contraction semigroup in $L^2(M, \mu)$. If in addition

(i) *$p(t, x, dy)$ is symmetric, that is*

$$\int_M \int_M u(x)v(y)p(t, x, dy)d\mu(x) = \int_M \int_M v(x)u(y)p(t, x, dy)d\mu(x) \quad \forall u, v \in C_c(M),$$

(ii) *there exists $D \subset L^2(M, \mu) \cap L^1(M, \mu)$ that is dense in $L^2(M, \mu)$ for which*

$$\lim_{t\to 0^+} \int_M u(y)p(t, x, dy) = u(x) \qquad \mu\text{-a.e. } x \in M,$$

then the semigroup $\{P_t\}_{t\geq 0}$ in (2.3.4) is strongly continuous.

Proof. For the second statement see [21, Proposition 4.3]. □

Going the other way around, that is from a strongly continuous contraction semigroup to a Markov process, we encounter the Markov property *for semigroups.*

Definition 2.3.5. A strongly continuous contraction semigroup $\{P_t\}_{t\geq 0}$ in $L^2(M, \mu)$ satisfies the Markov property if for any $u \in L^2(M, \mu)$ with $0 \leq u \leq 1$ μ-a.e. also $0 \leq P_t u \leq 1$ μ-a.e.

Theorem 2.3.6. *Let $\{P_t\}_{t\geq 0}$ be a strongly continuous contraction semigroup in $L^2(M,\mu)$ that is Markov. Then, there exists a Markov process $\{X_t\}_{t\geq 0}$ whose finite-dimensional distributions are uniquely determined by*

$$\begin{aligned}&\mathbb{P}_x(X_0\in A_0, X_{t_1}\in A_1,\ldots,X_{t_n}\in A_{t_n})\\&\quad=\int_{A_0}\cdots\int_{A_n} p(t_n-t_{n-1},y_{n-1},dy_n)\,p(t_{n-1}-t_{n-2},y_{n-2},dy_{n-1})\ldots p(t_1,y_0,dy_1)p(0,x,y_0),\end{aligned}$$

where $p(t,x,\cdot)$ is the transition kernel measure such that

$$P_tu(x)=\int_M u(y)p(t,x,dy)\qquad u\in L^2(M,\mu),\ \mu\text{-a.e. } x\in M. \tag{2.3.5}$$

Observation 2.3.7. The observant reader may already notice that if the semigroup comes from a heat kernel $\{p_t\}_{t\geq 0}$, that kernel is the density of the transition kernel measure in (2.3.5) with respect to the underlying measure, i.e.

$$p(t,x,dy)=p_t(x,y)d\mu(y).$$

Proof of Theorem 2.3.6. The existence of the transition kernel $p(t,x,dy)$ associated with $\{P_t\}_{t\geq 0}$ follows from an application of the so-called "bi-measure theorem"; a detailed proof can be found in [2, Proposition 1.2.3]. With the transition kernel on hand, since the underlying space M is complete and separable, the existence of $\{X_t\}_{t\geq 0}$ and its characterization through the finite-dimensional distributions is a consequence of Kolmogorov's theorem; see [7, Chapter 4, Theorem 1.1] and [3, p.16-17]. □

2.4. Markov semigroups and Dirichlet forms

In the previous section we have seen the Markov property for both stochastic processes and semigroups. Anyone wondering if there is such a property also for symmetric forms $(\mathcal{E},D(\mathcal{E}))$ in $L^2(M,\mu)$ will find the answer is yes, and that make it a Dirichlet form ☺.

Definition 2.4.1. A symmetric form $(\mathcal{E},D(\mathcal{E}))$ is said to be Markov if for any $u\in D(\mathcal{E})$ the function $\tilde{u}:=\min\{u_+,1\}\in D(\mathcal{E})$ and $\mathcal{E}(\tilde{u},\tilde{u})\leq\mathcal{E}(u,u)$.

Et finalement...

Definition 2.4.2. A closed Markov symmetric form on $L^2(M,\mu)$ is called a Dirichlet form.

In Theorem 2.2.1 we learned how to relate closed forms and strongly continuous contraction semigroups via

$$\mathcal{E}(u,v)=\lim_{t\to 0}\frac{1}{t}\langle u-P_tu,v\rangle.$$

The Markovian property will turn that statement into a relation between Markov semigroups and Dirichlet form.

Theorem 2.4.3. *A closed symmetric form $(\mathcal{E},D(\mathcal{E}))$ on $L^2(M,\mu)$ is Markov if and only if its associated strongly continuous contraction semigroup $\{P_t\}_{t\geq 0}$ is Markov.*

Proof. We prove here explicitly that $\{P_t\}_{t\geq 0}$ Markov implies $(\mathcal{E}, D(\mathcal{E}))$ is Markov, and refer the reader to [] for the converse (longer) direction.

Assume that $\{P_t\}_{t\geq 0}$ is Markov. Using the fact that the semigroup is conservative, i.e. $P_t\mathbf{1} = \mathbf{1}$, and symmetric, for any $u \in L^2(M,\mu)$ and $t \geq 0$ we have

$$\begin{aligned}
\langle u - P_t u\rangle &= \int_M u^2 d\mu - \int_M P_t u \cdot u\, d\mu \\
&= \int_M P_1\mathbf{1}u^2 d\mu - \int_M P_t u \cdot u\, d\mu \\
&= \int_M\int_M p_t(x,y)u^2(x)\, d\mu(y)\, d\mu(x) - \int_M\int_M p_t(x,y)u(y)u(x)\, d\mu(y)\, d\mu(x) \\
&= \frac{1}{2}\int_M\int_M p_t(x,y)u^2(x)\, d\mu(y)\, d\mu(x) + \frac{1}{2}p_t(x,y)u^2(y)\, d\mu(y)\, d\mu(x) \\
&- \int_M\int_M p_t(x,y)u(y)u(x)\, d\mu(y)\, d\mu(x) \\
&= \frac{1}{2}\int_M\int_M p_t(x,y)\big(u(x)-u(y)\big)^2 d\mu(y)d\mu(x).
\end{aligned}$$

Taking now $\tilde{u} := \min\{u_+, 1\}$, the latter computation implies

$$\frac{1}{t}\langle \tilde{u} - P_t\tilde{u}\rangle \leq \frac{1}{t}\langle u - P_t u\rangle$$

for any $t \geq 0$ and taking the limit as $t \to 0$ the Markovian property of $(\mathcal{E}, D(\mathcal{E}))$ follows. □

2.5. Dirichlet forms and energy measures

Recall that $C_c(M)$ denotes the space of continuous functions with compact support.

Definition 2.5.1. A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is regular if there exists $\mathcal{C} \subset C_c(M) \cap D(\mathcal{E})$ such that

(i) $\mathcal{C}$ is dense in $C_c(M)$ with respect to the supremum norm,

(ii) $\mathcal{C}$ is dense in $D(\mathcal{E})$ with respect to the norm

$$\|u\|_{\mathcal{E}_1} := \big(\mathcal{E}(u,u) + \|u\|_{L^2}\big)^{1/2}.$$

Theorem 2.5.2. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular Dirichlet form. For any $u \in D(\mathcal{E}) \cap L^\infty(M)$ there exists a unique Radon measure ν_u on M such that*

$$\mathcal{E}(u,v) = \frac{1}{2}\mathcal{E}(u^2,v) + \int_M v\, d\nu_u \qquad \forall\, v \in D(\mathcal{E}) \cap C_c(M). \tag{2.5.1}$$

Proof. Fix $u \in D(\mathcal{E}) \cap L^\infty(M)$ and define the functional

$$\begin{aligned}
F\colon C_c(M) &\longrightarrow \mathbb{R} \\
v &\longmapsto \mathcal{E}(v,uv) - \frac{1}{2}\mathcal{E}(v,u^2).
\end{aligned} \tag{2.5.2}$$

Because $(\mathcal{E}, D(\mathcal{E}))$ is bilinear, the functional F is linear. In addition, one can prove, see e.g. [8, p.122] that it satisfies

$$0 \leq F(v) \leq \|v\|_{L^\infty}\mathcal{E}(u,u)$$

and thus is in particular non-negative. By virtue of the Riesz-Markov-Kakutani representation theorem, see e.g. [23, Theorem 2.14] there exists a unique Radon measure, which we denote ν_u that satisfies (2.5.1). □

The measure ν_u from Theorem 2.5.2 is also called the *energy measure* of u because it is often the case that

$$\nu_u(M) = 2\mathcal{E}(u,u).$$

By polarization, energy measures give rise to the signed (!) Borel measure

$$\nu_{u,v} = \frac{1}{2}\big(\nu_{u+v} - \nu_u - \nu_v\big). \tag{2.5.3}$$

The interested reader is may find more details about energy measures and their properties in [8, Section 3.2].

One interesting geometric connection appears in [?], where energy measures are used to define an intrinsic pseudo metric on the underlying space M by

$$d_{\mathcal{E}}(x,y) := \sup\Big\{u(x) - u(y) \;:\; u \in D(\mathcal{E}) \cap C_c(M),\; \frac{d\nu_u}{d\mu} \leq 1\Big\}, \tag{2.5.4}$$

where $\frac{d\nu_u}{d\mu}$ denotes the Radon-Nykodym derivative of ν_u with respect to the underlying measure μ. When existent, that derivative plays the role of the square of the length of the gradient and corresponds to the an operator called *Carré du champ*. In fact, in the general context of metric measure spaces equipped with a Dirichlet form, the existence (or non-existence) of this derivative seems to be tightly connected with the Gaussian (or sub-Gaussian) nature of the associated heat kernel.

Definition 2.5.3. Let $(L, D(L))$ be an operator in $L^2(M,\mu)$ and let $\mathcal{A} \subset D(L)$ be a subspace with the property that $u{\cdot}v \in \mathcal{A}$ for any $u, v \in \mathcal{A}$. The bilinear map

$$\Gamma(u,v) := \frac{1}{2}\big(L(u{\cdot}v) - uLv - vLu\big) \qquad u, v \in \mathcal{A}, \tag{2.5.5}$$

is called the Carré du champ operator associated with $(L, D(L))$.

Proposition 2.5.4. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular Dirichlet form with infinitesimal generator $(L, D(L))$. Further, let $(\Gamma, \mathcal{A} \times \mathcal{A})$ be the Carré du champ operator associated with $(L, D(L))$. For any $u, v \in \mathcal{A} \cap L^\infty(M)$, $\Gamma(u,v)$ is the Radon-Nykodym derivative of the measure $\nu_{u,v}$ with respect to the underlying measure μ.*

Proof. Let $w \in \mathcal{A}$. On the one hand, applying the definition (2.5.5),

$$\begin{aligned}\int_M w\,\Gamma(u,v)\,d\mu &= \frac{1}{2}\langle L(uv), w\rangle - \frac{1}{2}\langle uLv, w\rangle - \frac{1}{2}\langle vLu, w\rangle \\ &= \frac{1}{2}\langle L(uv), w\rangle - \frac{1}{2}\langle Lv, wu\rangle - \frac{1}{2}\langle Lu, wv\rangle \\ &= -\frac{1}{2}\mathcal{E}(uv, w) + \frac{1}{2}\mathcal{E}(v, wu) + \frac{1}{2}\mathcal{E}(u, wv).\end{aligned}$$

On the other hand, by definition of the measure $\nu_{u,v}$ in (2.5.3),

$$\begin{aligned}
\int_M w\,\nu_{u,v} &= \frac{1}{2}\int_M w\,d\nu_{u+v} - \frac{1}{2}\int_M w\,d\nu_u - \frac{1}{2}\int_M w\,d\nu_v \\
&= \frac{1}{2}\mathcal{E}(u+v,(u+v)w) - \frac{1}{4}\mathcal{E}((u+v)^2,w) - \frac{1}{2}\mathcal{E}(u,uw) + \frac{1}{4}\mathcal{E}(u^2,w) \\
&- \frac{1}{2}\mathcal{E}(v,wu) + \frac{1}{4}\mathcal{E}(v^2,w) \\
&= \frac{1}{2}\mathcal{E}(u,uw) + \frac{1}{2}\mathcal{E}(u,vw) + \frac{1}{2}\mathcal{E}(v,uw) + \frac{1}{2}\mathcal{E}(v,vw) \\
&- \frac{1}{4}\mathcal{E}(u^2,w) - \frac{1}{4}\mathcal{E}(2uv,w) - \frac{1}{4}\mathcal{E}(v^2,w) \\
&- \frac{1}{2}\mathcal{E}(u,uw) + \frac{1}{4}\mathcal{E}(u^2,w) - \frac{1}{2}\mathcal{E}(v,wv) + \frac{1}{4}\mathcal{E}(v^2,w) \\
&= \frac{1}{2}\mathcal{E}(u,vw) + \frac{1}{2}\mathcal{E}(v,uw) - \frac{1}{2}\mathcal{E}(uv,w).
\end{aligned}$$

□

Definition 2.5.5. A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is called strictly local if the topology generated by the underlying measure μ is the same as the one generated by the pseudo metric $d_{\mathcal{E}}$.

CHAPTER 3

What is the use?

While as passionate learner the process of learning new concepts brings me joy (most of the time), I do need to find a personal motivation for learning them. Besides the fascination that connections between different areas and fields (within and outside of mathematics) awake in me, Dirichlet forms opened the door to the world of analysis and probability on fractals.

3.1. The heat equation on a fractal

In the previous chapters we have seen how the existence of a regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on a given locally compact metric measure space (M, μ, d) renders the ability to formulate an associated heat equation

$$\begin{cases} \partial_t u(t,x) = Lu(t,x), \quad x \in M \\ u(0,x) = u_0(x), \end{cases} \tag{3.1.1}$$

where L is the infinitesimal generator associated with $(\mathcal{E}, D(\mathcal{E}))$, and whose solution may be expressed in terms of an associated heat kernel $p_t(x,y)$ as

$$P_t u_0(x) = \int_M u_0(y)\, p_t(x,y)\, d\mu(y).$$

We have also learned that, alternatively we may write (3.1.1) weakly as

$$\begin{cases} \langle \partial_t u, \varphi \rangle = -\mathcal{E}(u, \varphi), & \forall\, \varphi \in C_c(M) \\ u(0,x) = u_0(x) & \mu\text{-a.e. } x \in M, \end{cases}$$

It may look like we can do this in *any* locally compact metric measure space (M, μ, d), however there is a fundamental question lying on the background, namely:

Is there a “natural” Markov process/Dirichlet form/Semigroup
associated with a space (M, μ, d)?

Not a trivial question, indeed. On fractals, answers started to appear in the late 80s with breakthrough probabilistic works on the Sierpinski gasket by Kusuoka [18] and Goldstein [9], on planar nested fractals by Lindstøm [20]. Pioneers in the analytic approach were Kigami [14] and Kusuoka [19], whose constructions applied to so-called post-critically finite self-similar sets. Constructions expanded to the Sierpinski carpet in a series of works by Barlow-Bass []. These were based on graph approximations of the (fractal) underlying space.

The following sections are meant to give context and provide overview of a particular method to construct Dirichlet forms in a family of metric measure spaces often called *Cheeger spaces*, which need not be approximated by a graph. Originally developed by Kumagai-Sturm in [17], my collaborator Fabrice Baudoin and I revisited this method in [1] to strengthen the probabilistic implications of their earlier results.

3.2. Lip-Poincaré inequalities

The underlying metric measure space (M, d, μ) will be a compact Cheeger space. To describe the latter we first need to talk briefly about Lipschitz functions and Poincaré inequalities. For more details we refer to [12, Chapter 6].

Definition 3.2.1. A function $u\colon M \to \mathbb{R}$ is C_L-Lipschitz if there exists a constant $C_L > 0$ such that

$$|u(x) - u(y)| \le C_L d(x, y)$$

for any $x, y \in M$. The Lipschitz constant of a Lipschitz function u is

$$(\operatorname{Lip} u)(y) := \limsup_{r\to 0^+} \sup_{x\in M, d(x,y)\le r} \frac{|u(x) - u(y)|}{r}.$$

We will denote the subspace of Lipschitz functions on M by $\operatorname{Lip}(M)$.

Definition 3.2.2. A metric measure space (M, d, μ) supports a (p, p)-Poincaré inequality with respect to Lipschitz functions if there exist constants $C > 0$ and $\lambda > 1$ such that

$$\int_B \Big|u - \fint_B u\, d\mu\Big|^p d\mu \le C(\operatorname{diam} B)^p \int_{\lambda B} (\operatorname{Lip} u)^p d\mu$$

for any ball $B \subset M$ and any $u \in \operatorname{Lip}(M)$. Here, $\fint_B u\, d\mu := \frac{1}{\mu(B)} \int_B u\, d\mu$.

To some extent, the scaling power p of the diameter is characteristic of this kind of spaces.

Definition 3.2.3. A locally compact metric measure space (M, d, μ) is called a Cheeger space if the following three properties are satisfied:

(i) The measure μ is a Radon measure, i.e., it is finite on compact sets, outer regular on Borel sets and inner regular on open sets.

(ii) The measure μ is doubling, i.e., the volume of a ball is bounded by the volume of a ball half its radius independently of the size of the radius. In other words, there exists a constant $C > 0$ such that for any $x \in M$ and $R > 0$,

$$\mu(B(x, 2R)) \le C\mu(B(x, R)). \tag{3.2.1}$$

(iii) The space satisfies a $(2,2)$-Poincaré inequality with respect to Lipschitz constants.

Remark 3.2.4. In the context of Cheeger spaces, the Lipschitz constant in the $(2,2)$-Poincaré inequality may equivalently be replaced by the *weak upper gradient* the function, i.e. to the inequality

$$\int_B |u-u_B|^p d\mu \le C(\operatorname{diam} B)^p \int_{\lambda B} (g_u)^2 d\mu,$$

c.f. [13, Theorem 8.4.2].

3.3. Mosco and Γ-convergence

The idea behind this way of constructing a "natural" or intrinsic Dirichlet form is to mimic a possible "natural" approximation of the underlying metric measure space (M,d,μ): If there is a sequence of meaningful approximations (M_n,d_n,μ_n) and one can construct in each of them a "natural" (Dirichlet) form, a natural candidate for a (Dirichlet) form in the limit should also be some sort of limit of forms.

What kind of limit?

This is what this section is about. There are two main convergence modes that involve functionals like Dirichlet forms: (de Giorgi) Γ-convergence and Mosco convergence. While the definitions presented here are for L^2-functionals, they are also meaningful in more general settings, see [6,22] and also [1].

We start with the older and weaker type of convergence, which is a standard took in calculus of variations.

Definition 3.3.1 (Γ-convergence)**.** A sequence of functionals $\{E_n\colon L^2(M,\mu)\to\mathbb{R}\}_{n\ge1}$ is said to Γ-converge to a functional $\mathcal{E}\colon L^2(M,\mu)\to\mathbb{R}$ if

(i) For any sequence $\{u_n\}_{n\ge1}\subset L^2(M,\mu)$ that converges *strongly* to $u\in L^2(X,\mu)$ in $L^2(M,\mu)$,

$$\liminf_{n\to\infty} E_n(u_n,u_n)\ge \mathcal{E}(u,u).$$

(ii) For any $u\in L^2(M,\mu)$ there exists a sequence $\{u_n\}_{n\ge1}\subset L^2(X,\mu)$ that converges strongly to u in $L^2(M,\mu)$ and

$$\limsup_{n\to\infty} E_n(u_n,u_n)\le \mathcal{E}(u,u).$$

In the context of Dirichlet forms, the stronger Mosco convergence mode became especially useful due to its implications in convergence of semigroups, resolvents and spectral families associated with the corresponding forms [22, Section 2]. The following definition, see [1], is a natural extension of Mosco's original definition to the functional setting.

Definition 3.3.2 (Mosco convergence)**.** A sequence of functionals $\{E_n\colon L^2(M,\mu)\to\mathbb{R}\}_{n\ge1}$ is said to Mosco-converge to a functional $\mathcal{E}\colon L^2(M,\mu)\to\mathbb{R}$ if

(i) For any sequence $\{u_n\}_{n\ge1}\subset L^2(M,\mu)$ that converges *weakly* to $u\in L^2(X,\mu)$ in $L^2(X,\mu)$,

$$\liminf_{n\to\infty} E_n(u_n,u_n)\ge \mathcal{E}(u,u).$$

(ii) For any $u \in L^2(M,\mu)$ there exists a sequence $\{u_n\}_{n\geq 1} \subset L^2(M,\mu)$ that converges strongly to u in $L^2(M,\mu)$ and

$$\limsup_{n\to\infty} E_n(u_n,u_n) \leq \mathcal{E}(u,u).$$

In both cases, the domain of the limiting form $\mathcal{E}$ is

$$D(\mathcal{E}) := \big\{u \in L^2(M,\mu), \mathcal{E}(u,u) < +\infty\big\}.$$

3.4. Korevaar-Schoen energy functionals

Why "Korevaar-Schoen"? Because of a seminal paper by these authors [15], where they developed a general theory of Sobolev spaces and harmonic maps between Riemannian manifolds based on the following type of functional.

Definition 3.4.1. For any $r>0$, a Korevaar-Schoen energy functional is a functional $E_r\colon L^2(M,\mu) \to \mathbb{R}$ of the form

$$E_r(f) := \frac{1}{r^2} \fint_{B(x,r)} |u(x)-u(y)|^2 d\mu(y)\, d\mu(x).$$

In the same spirit as the work by Korecaar and Schoen, these functionals have been studied in Cheeger spaces and compared with so-called Neutonian Sobolev spaces based on upper gradients. In particular, see [16], the Neutonian space $N^{1,2}(M)$ coincides with the Korevaar-Schoen space

$$KS^{1,2}(M) := \big\{u \in L^2(M,\mu)\colon \ \sup_{r>0} E_r(u) < \infty\big\},$$

and

$$\sup_{r>0} E_r(u) \simeq \|g_u\|_{L^2}$$

for any $u \in N^{1,2}(M)$.

3.5. Construction of Dirichlet forms

Lipschitz functions play a fundamental role in the structure of Cheeger spaces (M,d,μ) by connecting the geometry and the analysis on them. Indeed, the doubling property (3.2.1) implies the existence of a maximally separated ε-covering $\{B(x_i,\varepsilon)\}_{i\geq 1}$ of M with the bounded overlap property and a subordinated Lipschitz partition of unity $\{\varphi_i^\varepsilon\}_{i\geq 1}$. The interested reader may find in [13, pp. 102-104] a detailed account of these results. That covering and partition are key in proving the existence of a "natural" Dirichlet form on a Cheeger space (M,d,μ).

The Dirichlet form that will be constructed is *strictly local*, which means that the topology generated by the metric d is the same as the topology generated by the metric

$$d_{\mathcal{E}}(x,y) := \sup\{u(x)-u(y)\ :\ u \in D(\mathcal{E}) \cap C_c(M) \text{ and } d\Gamma(u,u) \ll d\mu\}.$$

Theorem 3.5.1. *On a compact Cheeger space (M,d,μ) there exists a strictly local and regular Dirchlet form $(\mathcal{E},D(\mathcal{E}))$ that is the Mosco limit of the Korevaar-Schoen type energies*

$$E_{r_n}(u) := \frac{1}{r_n^2} \fint_{B(x,r_n)} |u(x)-u(y)|^2 d\mu(y)\, d\mu(x),$$

where $r_n \to 0$ is independent of u.

Proof. Here we summarize the strategy and main ideas of the proof ideas and refer to [1] for rigorous arguments.

1. Show that there exists a constant $C > 0$ such that for any $u \in L^2(M,\mu)$ and $u_n \xrightarrow[n\to\infty]{L^2} u$

$$\sup_{r>0} E_r(u) \le C \liminf_{n\to\infty} E_{r_n}(u_n), \tag{3.5.1}$$

where $r_n \to 0$ is independent of C and u. The proof of (3.5.1) relies on approximations by Lipschitz functions: Any $u_n \in L^2(M,\mu)$ is approximated by

$$u_{n,\varepsilon} = \sum_{i\ge 0} \fint_{B(x_i,\varepsilon)} u_n \, d\mu \, \varphi_i^\varepsilon.$$

Applying the $(2,2)$-Poincaré inequality, the Lipschitz continuity of φ_i^ε and the bounded overlap property of the ε-covering one proves

$$\int_X (\operatorname{Lip} u_{n,\varepsilon})^2 d\mu \le C \frac{1}{\varepsilon^2} \int_M \fint_{B(x,2\varepsilon} |u_n(x) - u_n(y)|^2 d\mu(y)\, d\mu(x).$$

The latter is used to prove that for any $r > 0$

$$E_r(u_{n,\varepsilon_n}) \le C \liminf_{n\to\infty} E_{\varepsilon_n}(u_n).$$

2. The existence of a Γ-limit follows from a general result form Γ-convergence, which states that any sequence of functionals $\{E_{r_n}\}_{n\ge 0}$ contains a Γ-convergent subsequence [4, Theorem 8.5]. The form $(\mathcal{E}, D(\mathcal{E}))$ is defined to be the Γ-limit of that subsequence.

3. Upgrading Γ-convergence to Mosco convergence is possible when the underlying space is compact due to a result by Mosco [22] once one proves that the (sub-)sequence $\{E_{r_n}\}_{n\ge 0}$ is *asymptotically compact*, that is for any sequence $\{u_n\}_{n\ge 1} \subset L^2(M,\mu)$ for which

$$\liminf_{n\to\infty} \left(E_{r_n}(u_n) + \|u_n\|_{L^2}^2 \right) < \infty$$

has a subsequence that converges strongly in L^2. That result is classically called Rellich-Kondrachov and was proved in the Cheeger setting in [11, Theorem 8.1].

□

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