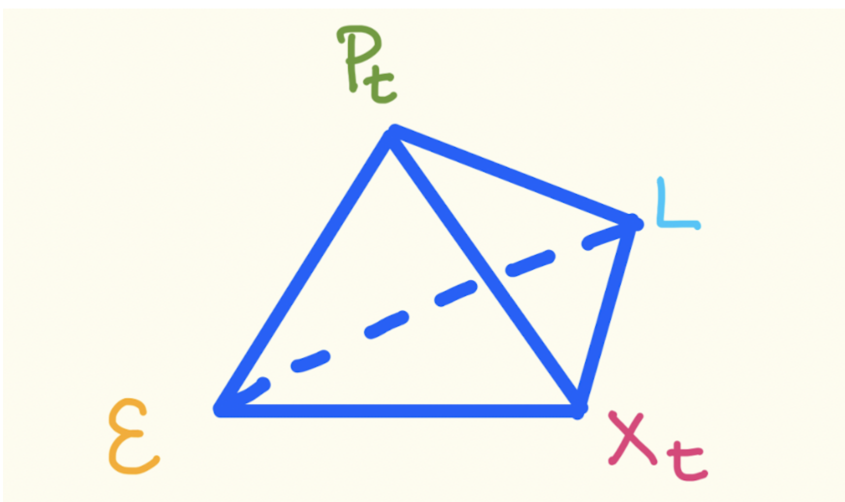
Semigroups, Markov processes and Dirichlet forms

Lecture notes

Patricia Alonso Ruiz



typed by class of Spring'23

Contents

CHAPTER 1

Semigroups, generators and Markov processes

Notation

Let us start with some notation that will be used throughout the chapter.

○ (M, d) is a metric space, $\mathcal{B}(M)$ its Borel σ-algebra;

○ $B := B(M, \mathbb{R})$ is the Banach space of bounded measurable functions $f \colon M \to \mathbb{R}$ with the supremum norm
$$\|f\| := \sup_{x \in M} |f(x)|;$$

○ B^* is the dual of B, i.e. bounded linear functionals $F \colon B \to \mathbb{R}$ with
$$\|F\| := \sup_{\|f\|=1} |F(f)|;$$

○ $C_b(M)$ is the Banach space of bounded continuous functions $f \in B$ with the supremum norm $\|f\|$;

○ For (M, d) is locally compact and separable, $C_0(M)$ denotes the Banach space of bounded continuous functions $f \in B$ that "vanish at infinity", i.e.
$$\forall \varepsilon > 0 \text{there exists } K \subset M \text{ compact such that } |f(x)| < \varepsilon \quad \forall x \in M \setminus K.$$

In addition the following convergence types will appear:

○ For a sequence $\{f_n\}_{n \geq 1} \subset B$ and $f \in B$:

- Strong convergence: $f_n \xrightarrow{s} f$ means $\|f_n - f\| \xrightarrow{n\to\infty} 0$ and may also be written as $f = s-\lim\limits_{n\to\infty} f_n$;
- Weak convergence: $f_n \xrightarrow{w} f$ means $|F(f_n) - F(f)| \xrightarrow{n\to\infty} 0$ for all $F \in B^*$ and may also be written as $f = w-\lim\limits_{n\to\infty} f_n$.

∘ For a sequence $\{Q_n\}_{n\geq 1}$ of bounded linear operators on B and a bounded linear operator Q on B:

- Uniform convergence: $Q_n \xrightarrow{u} Q$ means $\|Q_n - Q\| \xrightarrow{n\to\infty} 0$;
- Strong convergence: $Q_n \xrightarrow{s} Q$ means $\|Q_n f - Qf\| \xrightarrow{n\to\infty} 0$ for all $f \in B$;
- Weak convergence: $Q_n \xrightarrow{w} Q$ means $|F(Q_n f) - F(Qf)| \xrightarrow{n\to\infty} 0$ for all $F \in B^*$ and $f \in B$.

1.1. Contractive semigroups

We start with the basic property that defines a semigroup.

Definition 1.1.1. (i) A family of bounded linear operators $\{P_t\}_{t\geq 0}$ on B is called a *semigroup* if

$$P_0 = \mathrm{Id} \qquad \text{and} \qquad P_{s+t} = P_s P_t \quad \forall\, s,t \geq 0.$$

(ii) A semigroup $\{P_t\}_{t\geq 0}$ is called *contractive* if

$$\|P_t\| \leq 1 \quad \forall\, t \geq 0.$$

Definition 1.1.2. The *domain of strong continuity* of a contractive semigroup $\{P_t\}_{t\geq 0}$ on B is defined as

$$B_0 := \{f \in B\colon \lim_{t\to 0^+} \|P_t f - f\| = 0\}.$$

If $B = B_0$, then the semigroup is said to be *strongly continuous.*

The domain B_0 enjoys several useful properties.

Lemma 1.1.3. *Let $\{P_t\}_{t\geq 0}$ be a contractive semigroup on B. Then,*

(i) *B_0 is a Banach space.*

(ii) *For any $t \geq 0$, $P_t B_0 \subseteq B_0$.*

(iii) *$f \in B_0$ if and only if $P_{t+h} f \xrightarrow[h\to 0^+]{s} P_t f$ for all $t \geq 0$.*

The characterization (iii) is useful to check whether $B = B_0$.

Proof. (i) B_0 is a vector space (HW). To see B_0 is closed, let $\{f_n\}_{n\geq 0}\subset B_0$ such that $f_n \xrightarrow{s} f\in B$. We want to show that $f\in B_0$, i.e. $\lim_{t\to 0^+}\|P_tf-f\|=0$. Applying the triangle inequality and using the contractive property of P_t,

$$\begin{aligned}\|P_tf-f\| &\leq \|P_tf-P_tf_n\|+\|P_tf_n-f_n\|+\|f_n-f\|\\ &\leq \|P_t\|\|f-f_n\|+\|P_tf_n-f_n\|+\|f_n-f\|\\ &\leq 2\|f-f_n\|+\|P_tf_n-f_n\| \xrightarrow[t\to 0^+]{n\to\infty} 0\end{aligned}$$

(ii) Let $f\in B_0$ and $t\geq 0$. We want to prove $\lim_{h\to 0^+}\|P_hP_tf-P_tf\|=0$. By virtue of the semigroup property and its contractivity,

$$\|P_hP_tf-P_tf\|=\|P_{h+t}f-P_tf\|=\|P_t(P_hf-f)\|\leq\|P_t\|\|P_hf-f\|\leq\|P_hf-f\|.$$

Thus, $\limsup_{h\to 0^+}\|P_hP_t-P_tf\|=0$ and $P_tf\in B_0$.

(iii) $\Rightarrow$) follows from the computation above in (ii), $\Leftarrow$) follows by setting $t=0$. □

Where does one find semigroups? We will come back to this first example later!

Example 1.1.4. Let $B=C_b(\mathbb{Z})$, $\lambda>0$ and define for any $t\geq 0$

$$P_tf(x):=\sum_{k=0}^{\infty}f(x+k)\frac{(\lambda t)^k}{k!}e^{-\lambda t}.$$

$\{P_t\}_{t\geq 0}$ is a strongly continuous contractive semigroup.

- Well-defined and contractive: Using the series expression of the exponential function $e^{\lambda t}$,

$$\|P_tf\|\leq\|f\|\sum_{k=0}\frac{(\lambda t)^k}{k!}e^{-\lambda t}=\|f\|.$$

- Semigroup property: HW (algebra manipulations with the exponential series).

- Strong continuity:

$$\begin{aligned}\|P_tf-f\| &= \sup_{x\in\mathbb{Z}}\Big|\sum_{k=0}^{\infty}f(x+k)\frac{(\lambda t)^k}{k!}e^{-\lambda t}-f(x)\Big|\\ &= \sup_{x\in\mathbb{Z}}\Big|\sum_{k=1}^{\infty}f(x+k)\frac{(\lambda t)^k}{k!}e^{-\lambda t}+f(x)e^{-\lambda t}-f(x)\Big|\\ &\leq \|f\|\sum_{k=1}^{\infty}\frac{(\lambda t)^k}{k!}e^{-\lambda t}+(1-e^{-\lambda t})\|f\|=2(1-e^{-\lambda t})\|f\| \xrightarrow{t\to 0^+} 0.\end{aligned}$$

In general one can think of the semigroup property as somewhat being 'of exponential type'. This fact is reinforced by the next example.

Example 1.1.5. Let $L\colon B \to B$ be a bounded linear operator and define for any $t \geq 0$

$$P_t := e^{tL} := \sum_{k=0}^{\infty} \frac{t^k}{k!} L^k.$$

This is a strongly continuous semigroup.

- Well-defined: For any $n \geq 1$,

$$\Big\| \sum_{k=0}^{n} \frac{t^k}{k!} L^k \Big\| \leq \sum_{k=0}^{\infty} \frac{t^k}{k!} \|L^k\| \leq \sum_{k=0}^{\infty} \frac{t^k}{k!} \|L\|^k \leq e^{t\|L\|} < \infty.$$

- Semigroup: (HW)

- Strong continuity:

$$\|P_t - \mathrm{Id}\| = \Big\| \sum_{k=1}^{\infty} \frac{t^k}{k!} L^k \Big\| \leq \sum_{k=1}^{\infty} \frac{t^k}{k!} \|L\|^k = e^{t\|L\|} - 1 \xrightarrow{t \to 0^+} 0.$$

One of the most relevant examples is the semigroup associated with Brownian motion on $\mathbb{R}^d$.

Exercise 1.1.6. Let $B = C_b(\mathbb{R}^n)$ and define for any $t \geq 0$

$$P_t f(x) := \int_{\mathbb{R}^n} \frac{1}{(2\pi t)^{n/2}} e^{-\frac{\|x-y\|^2}{4t}} f(y)\, dy.$$

This is a continuous contractive semigroup.

We finish this first section with an a priori bound that puts again relevance in the exponential flavor of the semigroup property.

Lemma 1.1.7. *Let $\{P_t\}_{t \geq 0}$ be a strongly continuous semigroup on B. There exists $M \geq 1$ and $r \geq 0$ such that*

$$\|P_t\| \leq M e^{rt} \qquad \forall\, t \geq 0.$$

Proof. Assume first that there exists $M \geq 1$ such that $\|P_t\| \leq M$ for any $0 \leq t \leq t_0$ and some fixed $t_0 > 0$.
For any $t \geq 0$ we write now $t = kt_0 + s$ with $0 \leq s < t_0$ and $0 \leq k \in \mathbb{Z}$. Choosing $r = \frac{1}{t_0} \log M > 0$ we obtain from the semigroup property and our current assumption

$$\|P_t\| = \|P_{kt_0+s}\| = \|P_{kt_0} P_s\| \leq \|P_{kt_0}\| \|P_s\| \leq \|P_{t_0}^k\| M$$

$$\leq \|P_{t_0}\|^k M \leq M^k M = e^{rkt_0} M \leq M e^{rt}.$$

To prove the initial assumption we argue by contradiction: Suppose that there is no such $t_0 > 0$. Then, $\sup_{0<s\leq t_0} \|P_s\| = \infty$. By Banach-Steinhaus theorem, there exists $f \in B$ such that $\sup_{0\leq s\leq t_0} \|P_s f\| = \infty$ which contradicts the strong continuity of the semigroup. □

1.2. Infinitesimal Generators

We start this section by recalling some differentiation/integration properties of Banach space-valued functions. For a strongly continuous contractive semigroup $\{P_t\}_{t\geq 0}$ on B we define

◦ Differentiation: $\frac{d}{ds} P_s f := \lim_{h\to 0} \frac{1}{h}(P_{s+h} f - P_s f)$ for any $f \in B$.

◦ Integration: For $0 \leq a < b < \infty$ and consider a partition of $[a, b]$ given by $a = t_0 < t_1 < \ldots < t_n = b$ so that $\max\limits_{1\leq k<n} |t_k - t_{k-1}| \xrightarrow{n\to\infty} 0$. For any $f \in B$, we define

$$\int_b^a P_s f ds := s\text{–}\lim_{n\to\infty} \sum_{k=1}^n P_{s_k} f (t_k - t_{k-1}),$$

where $s_k \in [t_k, t_{k-1}]$ for every k, n.

We first state some useful properties of differentiation and integration for strongly continuous semigroup on B without proof.

Lemma 1.2.1. *Let $\{P_t\}_{t\leq 0}$ be a strongly continuous semigroup on B, then*

(i) $\left\|\int_b^a P_s f ds\right\| \leq \int_b^a \|P_s f\| ds$, *for all $f \in B$ and $s \geq 0$.*

(ii) $s\text{–}\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} P_s f ds = P_t f$, *for all $f \in B$ and $t \geq 0$.*

(iii) $P_t \left(\int_a^b P_s f ds\right) = \int_a^b P_{t+s} f ds = \int_{a+t}^{b+t} P_s f ds$, *for all $f \in B$, $0 \leq a < b < \infty$ and $t \geq 0$.*

(iv) $\int_b^a \frac{d}{ds} P_s f ds = P_b f - P_a f$, *for all $f \in B$, $0 \leq a < b < \infty$ and $s \geq 0$.*

We also introduce the main object we are going to study in this section.

Definition 1.2.2. The *infinitesimal generator* of a strongly continuous semigroup $\{P_t\}_{t\geq 0}$ on B is the linear operator

$$Lf := s\text{–}\lim_{t\to 0^+} \frac{1}{t}(P_t f - f). \tag{1.2.1}$$

We denote the domain of L by

$$D(L) = \{f \in B, \text{the limit in (1.2.1) exists}\}.$$

We will refer to the infinitesimal generator L as "*Laplacian*", since as we will see in the next Theorem, it satisfies the Heat equation (in (ii)).

Theorem 1.2.3. *Let $L : D(L) \to B$ be the infinitesimal generator of a strongly continuous contractive semigroup $\{P_t\}_{t\geq 0}$, then*

(i) $\overline{D(L)} = B_0$ *and* $L(D(L)) \subset B_0$*. In particular for all* $f \in B_0$ *and* $t \geq 0$*, we have*

$$\int_0^t P_s f ds \in D(L) \tag{1.2.2}$$

and

$$P_t f - f = L\left(\int_0^t P_s f ds\right). \tag{1.2.3}$$

(ii) *For any* $f \in D(L)$*, we have* $P_t f \in D(L)$ *and*

$$\frac{d}{dt} P_t f = L P_t f = P_t L f. \tag{1.2.4}$$

In particular

$$P_t f - f = \int_0^t L P_s f ds = \int_0^t P_s L f ds. \tag{1.2.5}$$

(iii) *L is closed.*

Proof. (i) $\overline{D(L)} = B_0$: We prove this in 2 steps: first prove that $D(L) \subset B_0$ then $B_0 \subset \overline{D(L)}$.

Given $\epsilon > 0$ and $f \in D(L)$, then by definition for any $\tilde{\varepsilon} > 0$ the exists a $\tilde{t_0} > 0$ such that $\|Lf - \frac{1}{t}(P_t f - f)\| < \tilde{\varepsilon}$ for all $t \leq \tilde{t_0}$. Then by inserting Lf and $\frac{1}{\tilde{t_0}}(P_{\tilde{t_0}} f - f)$, we have

$$\begin{aligned}
\|P_t f - f\| &= t\|\frac{1}{t}(P_t f - f) - Lf + Lf - \frac{1}{\tilde{t_0}}(P_{\tilde{t_0}} f - f) + \frac{1}{\tilde{t_0}}(P_{\tilde{t_0}} f - f)\| \\
&\leq t\|\frac{1}{t}(P_t f - f) - Lf\| + t\|Lf - \frac{1}{\tilde{t_0}}(P_{\tilde{t_0}} f - f)\| + t\|\frac{1}{\tilde{t_0}}(P_{\tilde{t_0}} f - f)\| \\
&\leq t\left(\tilde{\varepsilon} + \tilde{\varepsilon} + \frac{1}{\tilde{t_0}}(\|P_{\tilde{t_0}}\| \cdot \|f\| + \|f\|\right) \\
&\leq 2t\left(\tilde{\varepsilon} + \frac{1}{\tilde{t_0}}\|f\|\right),
\end{aligned}$$

where we use contractivity of $\{P_t\}$ at the last line. Then by choosing $t_0 = \min\{\tilde{t_0}, \dfrac{\varepsilon}{\tilde{\varepsilon} + \frac{1}{\tilde{t_0}}\|f\|}\}$, we have $\|P_t f - f\| < \varepsilon$.

To prove the second part, we first observe that, by Lemma 1.2.1 (ii), (iii),

$$\begin{aligned}
P_tf - f &= P_tf - P_0f \\
&= s\text{–}\lim_{h\to 0}\frac{1}{h}\left[\int_t^{t+h} P_sf\,ds - \int_0^h P_sf\,ds\right] \\
&= s\text{–}\lim_{h\to 0}\frac{1}{h}\left[\int_h^{t+h} P_sf\,ds - \int_0^t P_sf\,ds\right] \quad \left(\text{insert } \int_h^t P_sf\,ds\right) \\
&= s\text{–}\lim_{h\to 0}\frac{1}{h}\left[\int_0^{t} P_{h+s}f\,ds - \int_0^t P_sf\,ds\right] \\
&= s\text{–}\lim_{h\to 0}\frac{1}{h}(P_h - I)\left[\int_0^t P_sf\,ds\right] \\
&= L\int_0^t P_sf\,ds.
\end{aligned}$$

Since the equality holds for all fixed t, we see that limit in the second last line exists and last line follows by definition of L, which proves (1.2.2) and implies (1.2.3) and $B_0 \subset \overline{D(L)}$ by choosing the sequence $\{f_n\} = \{n\int_0^{1/n} P_sf\,ds\} \subset D(L)$.
Next if $f \in D(L) \subset B_0$, then, by Lemma 1.1.3, since $\frac{1}{t}(P_tf - f) \in B_0$ and B_0 is complete, we have $Lf \in B_0$.

(ii) Second equality of (1.2.4) follows from continuity of $\{P_t\}$. To prove the first equality, we notice that one side of the limit follows by definition of L, namely

$$s\text{–}\lim_{h\to 0^+}\frac{1}{h}(P_hP_tf - P_tf) = LP_tf,$$

since $P_tf \in D(L)$. For the other side, by replacing h by $-h$, we have

$$s\text{–}\lim_{h\to 0^-}\frac{1}{h}(P_{t+h}f - P_tf) = s\text{–}\lim_{h\to 0^+}\frac{1}{h}(P_tf - P_{t-h}f).$$

Then

$$\begin{aligned}
&\Big\|\frac{1}{h}(P_tf - P_{t-h}f) - LP_tf\Big\| \\
&= \Big\|\frac{1}{h}(P_{t-h}P_hf - P_{t-h}f) - P_tLf\Big\| \qquad \text{(second equality of (1.2.4))} \\
&\le \big\|P_{t-h}\big\|\Big\|\frac{1}{h}(P_hf - f) - P_hLf\Big\| \\
&\le \Big\|\frac{1}{h}(P_hf - f) - Lf\Big\| + \big\|Lf - P_hLf\big\| \quad (\|P_{t-h}\| \le 1) \\
&\xrightarrow{h\to 0^+} 0 + 0,
\end{aligned}$$

where the last line follows by definition of L and strong continuity of P_t. This proves (1.2.4).
By Lemma 1.2.1 (iii), definition of L and (1.2.3), we have

$$
\begin{aligned}
&\left\| P_t f - f - \int_0^t L P_s f ds \right\| \\
&\leq \left\| P_t f - f - \frac{1}{h}(P_h - I) \int_0^t P_s f ds \right\| \quad \text{(vanishes in limit by (1.2.3))} \\
&+ \left\| \frac{1}{h}(P_h - I) \int_0^t P_s f ds - \int_0^t L P_s f ds \right\| \\
&\leq \left\| \int_0^t P_s (\frac{1}{h}(P_h - I) f - L f) ds \right\| + O(h) \quad \text{(by Lemma 1.2.1 (iii) and (1.2.3))} \\
&\leq \int_0^t \| \frac{1}{h}(P_h - I) f - L f \| ds + O(h) \xrightarrow{h \to 0^+} 0, \quad \text{(by definition of } L\text{)}
\end{aligned}
$$

which proves the first equality of (1.2.5). Second equality follows by second equality of (1.2.4).

(iii) To prove that L is closed, assuming that $\{f_n\} \subset D(L)$, such that $f_n \xrightarrow{s} f$ and $L f_n \xrightarrow{s} g$ for some $f, g \in B$, we want to prove $Lf = g$.
First observe that

$$
\begin{aligned}
&\left\| P_h f - f - \int_0^h P_s g ds \right\| \\
&\leq \left\| P_h (f - f_n) + (P_h f_n - f_n) + (f_n - f) - \int_0^h P_s g ds \right\| \\
&\leq (\|P_h\| + 1) \|f - f_n\| + \left\| P_h f_n - f_n - \int_0^h P_s g ds \right\| \\
&\leq 2\|f - f_n\| + \left\| P_h f_n - f_n - \int_0^h P_s g ds \right\| \\
&= 2\|f - f_n\| + \left\| \int_0^h P_s L f_n ds - \int_0^h P_s g ds \right\| \quad \text{(by (1.2.5))} \\
&\leq 2\|f - f_n\| + \int_0^h \|P_s\| \|L f_n - g\| ds \\
&\xrightarrow{n \to \infty} 0 + 0. \quad \text{(since } f_n \xrightarrow{s} f \text{ and } L f_n \xrightarrow{s} g\text{)}
\end{aligned}
$$

Then by Lemma 1.2.1 (ii) and definition of L,

$$
g = s\text{-}\lim_{h \to 0^+} \frac{1}{h} \int_0^h P_s g ds = s\text{-}\lim_{h \to 0^+} \frac{1}{h}(P_h f - f) = Lf.
$$

Hence L is closed.

□

Example 1.2.4. Let A be a linear bounded operator on B and define $\{P_t\}$ as $P_t = e^{tA} = \sum\limits_{k=0}^{\infty} \frac{t^k}{k!}A^k$. Let's find out the infinitesimal operator through a priori calculation.

$$
\begin{aligned}
&\|\frac{1}{t}(P_t - I) - L\| \\
&= \|\frac{1}{t}\sum_{k=1}^{\infty}\frac{t^k}{k!}A^k - L\| = \|A + \frac{1}{t}\sum_{k=2}^{\infty}\frac{t^k}{k!}A^k - L\| \\
&\leq \frac{1}{t}\sum_{k=2}^{\infty}\frac{t^k}{k!}\|A\|^k + \|A - L\| \\
&\leq \frac{1}{t}(e^{t\|A\|} - 1 - t\|A\|) + \|A - L\|,
\end{aligned}
$$

where the first term in the last line vanishes as $t \to 0^+$ so if we choose $L = A$ we have $L = s\!-\!\lim\limits_{t\to 0^+} \frac{1}{t}(P_t - I) = A$.

Example 1.2.5. Let $B = C_b(\mathbb{Z})$ and, for $f \in B$ and $\lambda > 0$, define $\{P_t\}$ by

$$
P_t f(x) = \sum_{k=0}^{\infty} f(x+k)\frac{(\lambda t)^k}{k!}e^{-\lambda t}.
$$

We showed in Example 1.1.4 that this is a strongly continuous contractive semi-group. To find the infinitesimal operator, as in the previous example, we first do a priori calculation. For $f \in B$,

$$
\begin{aligned}
&\sup_{x\in\mathbb{Z}}\left|\frac{1}{t}(P_t f(x) - f(x)) - Lf(x)\right| \\
&= \sup_{x\in\mathbb{Z}}\left|\frac{1}{t}\left(f(x)e^{-\lambda t} + f(x+1)(\lambda t)e^{-\lambda t} + \sum_{k=2}^{\infty} f(x+k)\frac{(\lambda t)^k}{k!}e^{-\lambda t} - f(x)\right) - Lf(x)\right| \\
&= \sup_{x\in\mathbb{Z}}\left|\frac{1}{t}\sum_{k=2}^{\infty} f(x+k)\frac{(\lambda t)^k}{k!}e^{-\lambda t} + f(x)\frac{e^{-\lambda t}-1}{t} + f(x+1)\lambda e^{-\lambda t} - Lf(x)\right| \\
&\leq \frac{1}{t}(e^{-\lambda t} - 1 - \lambda t)\|f\| + \sup_{x\in\mathbb{Z}}\left|f(x)\frac{e^{-\lambda t}-1}{t} + f(x+1)\lambda e^{-\lambda t} - Lf(x)\right|,
\end{aligned}
$$

in which the first term vanishes as $t \to 0^+$ just as before. For the second term, since the limit

$$
Lf(x) = \lim_{t\to 0^+} f(x)\frac{e^{-\lambda t}-1}{t} + f(x+1)\lambda e^{-\lambda t} = -\lambda f(x) + \lambda f(x+1)
$$

exists. We have $L = -\lambda I + \lambda T$, where I is identity operator and T is the right shift operator ($Tf(x) = f(x+1)$).

Example 1.2.6. Let $B = C_0(\mathbb{R}^n)$ and for $f \in B$ define $\{P_t^{BM}\}$ by

$$P_t^{BM} f(x) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x-y|^2}{4t}} f(y)dy,$$

which forms strongly continuous contractive semigroup on B. Then after some complicated calculation you can show that for $f \in C_0^2(\mathbb{R}^n)$

$$\frac{1}{t}(P_t^{BM} f(x) - f(x)) = \sum_{i=1}^{n} \partial_i^2 f(x) + \sum_{i,j=1}^{n} R_{ij}(x,t),$$

where R_{ij} are remainder terms that converges to zero as $t \to 0$ uniformly in x. Then by density of $C_0^2(\mathbb{R}^n)$ in $C_0(\mathbb{R}^n)$, we see that L is Laplacian $\Delta = \sum_{i=1}^{n} \partial_i^2$. Note that $\{P_t^{BM}\}$ is not strongly continuous if we take $B = C_b(\mathbb{R}^n)$, for they are not even weakly continuous in $C_b(\mathbb{R}^n)$.

1.3. Resolvent

Definition 1.3.1. Let $A : B \to B$ be a closed operator.

(i) The *resolvent set of* A is the subset of $\mathbb{R}$ defined as

$$\rho(A) := \{\lambda \in \mathbb{R} : (\lambda I - A)^{-1} \text{ is a bounded operator}\}.$$

(ii) The *spectrum of* A is the closed set

$$\sigma(A) := \mathbb{R} \backslash \rho(A).$$

Strictly speaking, when we say $A : B \to B$ is a closed operator, we actually mean $A : D(A) \subset B \to B$ is closed, i.e. if $\{f_n\} \subset D(A)$, with $f_n \xrightarrow{n\to\infty} f \in D(A)$ and $Af_n \xrightarrow{n\to\infty} g$, then $Af = g$.

Observation 1.3.2. For $A : B \to B$ linear operator (not necessarily closed), we can define the resolvent by

$\{\lambda \in \mathbb{R} : (\lambda I - A)$ bijective, $(\lambda I - A)^{-1}$ continuous on B and $D((\lambda I - A)^{-1}) = B\}$.

Moreover, if $\rho(A) \neq \emptyset$ then A is closed. The resolvent set is open if A closed (HW).

Definition 1.3.3. Let $A : B \to B$ be a closed operator. For each $\lambda > 0$, the operator $R_\lambda := (\lambda I - A)^{-1}$ is called *the resolvent of A (at λ).*

Lemma 1.3.4. *Let $A : B \to B$ be a closed operator.*

(i) $R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$ *(Resolvent Identity).*
In particular, $R_\lambda R_\mu = R_\mu R_\lambda$.

(ii) *For every* $\lambda, \mu \in \rho(A)$, $R_\lambda B = R_\mu B = D(A)$.

(iii) *If* $(0,\infty) \subset \rho(A)$ *and* $\|\lambda R_\lambda\| \leq 1$ $\forall \lambda > 0$, *then*

$$\overline{R_\lambda B} = \{f \in B : \|\lambda R_\lambda f - f\| \xrightarrow{\lambda\to\infty} 0\}.$$

Proof. (i) We have

$$\begin{aligned} R_\lambda &= R_\lambda R_\mu^{-1} R_\mu = R_\lambda(\mu I - A)R_\mu = R_\lambda((\mu-\lambda)I + \lambda I - A)R_\mu \\ &= R_\lambda((\mu-\lambda)I + R_\lambda^{-1})R_\mu = (\mu-\lambda)R_\lambda R_\mu + R_\mu, \end{aligned}$$

which gives us the result. In particular, using this identity for $\mu \neq \lambda$ we get

$$R_\lambda R_\mu = (\mu-\lambda)^{-1}(R_\lambda - R_\mu) = (\lambda-\mu)^{-1}(R_\mu - R_\lambda) = R_\mu R_\lambda,$$

where the second equality follows by interchanging the roles of μ and λ.

(ii) We show first that $R_\lambda B = R_\mu B$. Notice that it is enough to show $R_\lambda B \subset R_\mu B$, since by interchanging the roles of μ and λ, we get $R_\mu B \subset R_\lambda B$. To see this, let $f \in R_\lambda B$, i.e. $f = R_\lambda g$ for some $g \in B$. We want $f = R_\mu h$ for some $h \in B$. Using the Resolvent Identity, we can write

$$f = R_\lambda g = (\mu-\lambda)R_\lambda R_\mu g + R_\mu g = R_\mu((\mu-\lambda)R_\lambda g + g) = R_\mu h,$$

for $h = (\mu-\lambda)R_\lambda g + g$.

To show $R_\lambda B \subset D(A)$, let $f = R_\lambda g$ for some $g \in B$. Then,

$$Af = (A - \lambda I)f + \lambda I f = R_\lambda^{-1} f + \lambda I f = g + \lambda R_\lambda g \in B.$$

For the reverse inclusion, $D(A) \subset R_\lambda B$, let $f \in D(A)$. We want $f = R_\lambda g$ for some $g \in B$. We have

$$f = R_\lambda R_\lambda^{-1} f = R_\lambda(\lambda I - A)f = R_\lambda g,$$

where $g = (\lambda I - A)f \in B$.

(iii) We first show that $R_\lambda B \subset \{f \in B : \|\lambda R_\lambda f - f\| \xrightarrow{\lambda\to\infty} 0\}$. Let $f = R_\lambda g$ for some $g \in B$. Let also $\mu > \lambda$. Then, using the resolvent identity, we have

$$\begin{aligned} \mu R_\mu f - f = \mu R_\mu R_\lambda g - R_\lambda g &= \frac{\mu}{\lambda-\mu}(R_\mu g - R_\lambda g) - R_\lambda g \\ &= \frac{\lambda}{\mu-\lambda} R_\lambda g - \frac{\mu}{\mu-\lambda} R_\mu g. \end{aligned}$$

Therefore, since $\|\lambda R_\lambda\| \leq 1$, $\forall \lambda > 0$

$$\|\mu R_\mu f - f\| \leq \frac{1}{\mu-\lambda}\|\lambda R_\lambda g\| + \frac{1}{\mu-\lambda}\|\mu R_\mu g\| \leq 2\frac{1}{\mu-\lambda}\|g\|$$

So, taking lim sup in both sides, we have

$$\limsup_{\mu\to\infty} \|\mu R_\mu f - f\| \leq \limsup_{\mu\to\infty} \frac{2}{\mu-\lambda}\|g\| = 0,$$

and we get the result.

To show that $\overline{R_\lambda B} \subset \{f \in B : \|\lambda R_\lambda f - f\| \xrightarrow{\lambda\to\infty} 0\}$, let $\{f_n\}_{n\geq 1} \subset R_\lambda B$ such that $\|f_n - f\| \xrightarrow{n\to\infty} 0$. Then, using the assumption that $\|\lambda R_\lambda\| \leq 1$, $\forall \lambda > 0$, we have

$$\begin{aligned}\|\lambda R_\lambda f - f\| &\leq \|\lambda R_\lambda f - \lambda R_\lambda f_n\| + \|\lambda R_\lambda f_n - f_n\| + \|f_n - f\| \\ &\leq \|\lambda R_\lambda\|\|f - f_f n\| + \|\lambda R_\lambda f_n - f_n\| + \|f_n - f\| \\ &\leq 2\|f - f_n\| + \| + \|\lambda R_\lambda f_n - f_n\|\end{aligned}$$

which goes to zero as $\lambda \to \infty$ and $n \to \infty$ and we get the result.

□

Definition 1.3.5. Let $\{P_t\}_{t\geq 0}$ be a contractive semigroup on B with infinitesimal generator L. We define the *resolvent of* $\{P_t\}_{t\geq 0}$ on B_0 to be the resolvent of L.

Theorem 1.3.6. *Let $\{P_t\}_{t\geq 0}$ be a contractive semigroup on B with infinitesimal generator L. Then,*

(i) *The resolvent of $\{P_t\}_{t\geq 0}$ exists for any $\lambda > 0$, and equals*

$$R_\lambda f = \int_0^\infty e^{-\lambda s} P_s f\, ds =: \mathcal{L}_\lambda f, \qquad \forall f \in B.$$

(ii) $\overline{R_\lambda B_0} = B_0$*,* $\forall \lambda \in \rho(L)$*.*

Proof. (i) We show first that $\mathcal{L}_\lambda$ is a bounded operator. Indeed, since $\{P_t\}_{t\geq 0}$ is contractive, for each $f \in B$, we have

$$\|\mathcal{L}_\lambda f\| \leq \int_0^\infty e^{-\lambda s}\|P_s f\| ds \leq \int_0^\infty e^{-\lambda s}\|f\| ds = \frac{1}{\lambda}\|f\|.$$

It remains to show that $\mathcal{L}_\lambda = (\lambda I - L)^{-1}$. In particular, for the right inverse, let $f \in B$, and we need to show

$$f = (\lambda I - L)\mathcal{L}_\lambda f = (\lambda I - L) \lim_{t\to\infty} \int_0^t e^{-\lambda s} P_s f\, ds.$$

It is left as an exercise to show that $P_s^\lambda f := e^{-\lambda s} P_s f$ defines a strongly continuous, contractive semigroup with infinitesimal generator $L_\lambda := L - \lambda I$. Therefore, we have

$$(\lambda I - L) \int_0^t e^{-\lambda s} P_s f\, ds = -L_\lambda \int_0^t e^{-\lambda s} P_s f\, ds = f - P_t^\lambda f = f - e^{-\lambda t} P_t f, \quad (1.3.1)$$

where the second equality follows from Theorem 1.2.3 (i). Now, since

$$\|e^{-\lambda t}P_t f\| \leq e^{-\lambda t}\|f\| \xrightarrow{t\to\infty} 0,$$

Taking limit as $t \to \infty$ in (1.3.1) we get the result.

For the left inverse, let $f \in B$, and we need to show

$$f = \mathcal{L}_\lambda(\lambda I - L)f.$$

We have

$$\begin{aligned}
\mathcal{L}_\lambda Lf &= \int_0^\infty e^{-\lambda s}P_s Lf\, ds = \int_0^\infty e^{-\lambda s}LP_s f\, ds = \int_0^\infty e^{-\lambda s}\frac{d}{ds}P_s f\, ds \\
&= e^{-\lambda s}P_s f\Big]_0^\infty + \lambda\int_0^\infty e^{-\lambda s}P_s f\, ds = -f + \lambda\mathcal{L}_\lambda f,
\end{aligned}$$

where the second and third equalities follow from Theorem 1.2.3 and the next comes from an integration by parts. Rearranging the terms above we get the result.

(ii) We first show that $R_\lambda B_0 \subset B_0$. To this end, let $f = R_\lambda g \in R_\lambda B_0$. Then

$$P_t f = P_t R_\lambda g = \int_0^\infty e^{-\lambda s}P_{t+s}g ds$$

by part (i) of this theorem. Then with the change of variables, $\tilde{t} = t + s$, we have

$$\begin{aligned}
P_t f &= \int_t^\infty e^{-\lambda(\tilde{t}-t)}P_{\tilde{t}}g d\tilde{t} = e^{\lambda t}\left[\int_0^\infty e^{-\lambda\tilde{t}}P_{\tilde{t}}g d\tilde{t} - \int_0^t e^{-\lambda\tilde{t}}P_{\tilde{t}}g d\tilde{t}\right] \\
&= e^{\lambda t}\left[R_\lambda g - \int_0^t e^{-\lambda\tilde{t}}P_{\tilde{t}}g d\tilde{t}\right] \xrightarrow{t\downarrow 0} f - 0 = f.
\end{aligned}$$

Hence, $f \in B_0$ and $R_\lambda B_0 \subset B_0$. Since B_0 is closed, this implies that in fact, $\overline{R_\lambda B_0} \subseteq B_0$.

Next, we show $B_0 \subset \overline{R_\lambda B_0}$. Let $f \in B_0$ and $f_n := n\int_0^\infty e^{-sn}P_s f ds \in R_n B_0$. Then we have

$$\begin{aligned}
\|f_n - f\| &= \left\|n\int_0^\infty e^{-sn}P_s f ds - n\int_0^\infty e^{-sn}f ds\right\| \leq n\int_0^\infty e^{-sn}\|P_s f - f\| ds \\
&= \int_0^\infty e^{-\tilde{s}}\|P_{\tilde{s}/n}f - f\| d\tilde{s} \xrightarrow{n\to\infty} 0
\end{aligned}$$

because $\|P_{\tilde{s}/n}f - f\| \xrightarrow{n\to\infty} 0$ since $f \in B_0$. So $f_n \xrightarrow{n\to\infty} f$, $f \in \overline{R_\lambda B_0}$, and hence $B_0 \subseteq \overline{R_\lambda B_0}$.

□

Exercise 1.3.7. Show the resolvent of $\left\{P_t^{BM}\right\}_{t\geq 0}$ on $C_0(\mathbb{R})$ is

$$R_\lambda f(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|x-y|} f(y)dy.$$

Example 1.3.8. Let us show that the domain of the infinitesimal generator L of $\left\{P_t^{BM}\right\}_{t\geq 0}$ on $C_0(\mathbb{R})$ is

$$C_0^2(\mathbb{R}) = \left\{f \in C_0(\mathbb{R}) \cap C^2(\mathbb{R}) : f'' \in C_0(\mathbb{R})\right\}.$$

First, let $f \in D(L)$. Then we can write

$$\begin{aligned}
f = R_\lambda g &= \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}x} \int_{-\infty}^{x} e^{\sqrt{2\lambda}y} g(y)dy + \frac{1}{\sqrt{2\lambda}} e^{\sqrt{2\lambda}x} \int_{x}^{\infty} e^{-\sqrt{2\lambda}y} g(y)dy \in C_0(\mathbb{R}) \\
f' &= -e^{-\sqrt{2\lambda}x} \int_{-\infty}^{x} e^{\sqrt{2\lambda}y} g(y)dy + e^{\sqrt{2\lambda}x} \int_{x}^{\infty} e^{-\sqrt{2\lambda}y} g(y)dy \in C_0(\mathbb{R}) \\
f'' &= 2\lambda R_\lambda g - 2g \in C_0(\mathbb{R})
\end{aligned}$$

so $f \in C_0^2(\mathbb{R})$ and $D(L) \subset C_0^2(\mathbb{R})$. The details of the equality for f'' above are left as a homework exercise.
Also notice that we have

$$f'' = 2\lambda f - 2R_\lambda^{-1} f = 2\lambda f - 2\lambda f + 2Lf = 2Lf \tag{1.3.2}$$

so we see again that $Lf = \frac{1}{2} f''$.
Next, we show that $C_0^2(\mathbb{R}) \subset D(L)$. The proof will be left as a homework exercise, but we will give the general idea here. Let $f \in C_0^2(\mathbb{R})$. We would like to show there exists g such that $f = R_\lambda g$, i.e. $gR_\lambda^{-1} f$. Based on equation (1.3.2), we can define $h = \lambda f - \frac{1}{2} f''$ (guess for g). Then we write the differential equation satisfied by $u = g - h$ and check it has only solution $u = 0$, whence $g = h$.

Corollary 1.3.9. *Let $\{P_t\}_{t\geq 0}$ be a strongly continuous contractive semigroup on B with infinitesimal generator L. Then*

$$\|(\lambda I - L)f\| \geq \lambda \|f\|, \quad \forall f \in B. \tag{1.3.3}$$

(Note: an operator satisfying (1.3.3) is called dissipative)

Proof. By Theorem 1.3.6 and assumption, $\overline{R_\lambda B_0} = B_0 = B$. Let $f \in R_\lambda B_0$. Then

$$\|f\| = \|R_\lambda g\| = \frac{1}{\lambda} \|\lambda R_\lambda g\| \leq \frac{1}{\lambda} \|g\| = \frac{1}{\lambda} \|(\lambda I - L)f\|.$$

□

Proposition 1.3.10. *Let $A : D(A) \subset B \to B$ be a linear operator. Assume that*

$$t \mapsto u(t) \in D(A), \quad \textit{and} \tag{1.3.4}$$

$$t \mapsto Au(t) \in B \tag{1.3.5}$$

are continuous functions for all $t > 0$, and for all $\varepsilon > 0$,

$$u(t) = u(\varepsilon) + \int_{\varepsilon}^{t} Au(s)ds \tag{1.3.6}$$

for all $t > \varepsilon$. Then $\|u(t)\| \le \|u(0)\|$ for all $t \ge 0$.

Proof. (Homework) □

Lemma 1.3.11. *A strongly continuous contractive semigroup is uniquely determined by its infinitesimal generator.*

Proof. Suppose $\{P_t\}_{t\ge0}$ and $\{Q_t\}_{t\ge0}$ are strongly contractive semigroups both with infinitesimal generator L. We would like to apply Proposition 1.3.10, with $u(t) = P_t - Q_t$ and $A = L$, so we must check the conditions (1.3.4), (1.3.5), and (1.3.6). Condition (1.3.4) holds since P_t and Q_t are continuous in t. By Theorem 1.2.3, $\overline{D(L)} = B_0$, so (1.3.5) holds. To see that (1.3.6) holds, notice that for $t > \varepsilon > 0$,

$$(P_t - Q_t)f - (P_\varepsilon - Q_\varepsilon)f = \int_{\varepsilon}^{t} \frac{d}{ds}(P_s - Q_s)f ds = \int_{\varepsilon}^{t} L(P_s - Q_s)ds$$

where the last equality comes from the heat equation. Hence, we may apply Proposition 1.3.10, and we have $\|P_t - Q_t\| \le \|P_0 - Q_0\| = 0$ for all $t \ge 0$, so $P_t = Q_t$ is uniquely determined by L. □

1.4. Hille-Yosida Theorem

Definition 1.4.1. Let $A : D(A) \subset B \to B$ be a closed linear operator with $\overline{D(A)} = B$ and $(0, \infty) \subset \rho(A)$. For each $\lambda \in \rho(A)$,

$$A_\lambda f := \lambda A R_\lambda f, \quad f \in B,$$

where R_λ is the resolvent of A, is called the Yosida approximation of A.

Some useful identities are given by

$$\lambda A R_\lambda f = \lambda(A - \lambda I + \lambda I)R_\lambda f = -\lambda f + \lambda^2 R_\lambda f \tag{1.4.1}$$

$$= -\lambda R_\lambda R_\lambda^{-1} f + \lambda^2 R_\lambda f = \lambda R_\lambda(-R_\lambda^{-1} + \lambda I)f. \tag{1.4.2}$$

Note that (1.4.2) is valid as long as $f \in D(A)$.

Proposition 1.4.2. *Let $A : D(A) \subset B \to B$ be a closed linear operator with $\overline{D(A)} = B$ and $(0, \infty) \subset \rho(A)$. If $\|\lambda R_\lambda\| \leq 1$, then the Yosida approximation of A satisfies:*

(i) A_λ *is bounded or all* $\lambda \in \rho(A)$

(ii) $A_\lambda A_\mu = A_\mu A_\lambda$ *for all* $\lambda, \mu \in \rho(A)$

(iii) $Af = \operatorname{s-lim}_{n\to\infty} A_\lambda f$ *for all* $f \in D(A)$

Proof. (i) Using (1.4.1), $\|A_\lambda f\| = \|\lambda A R_\lambda f\| = \|\lambda^2 R_\lambda f - \lambda f\| \leq \lambda\|\lambda R_\lambda f\| + \lambda\|f\| \leq 2\lambda\|f\|$

(ii) Using (1.4.1) again, $A_\lambda A_\mu f = (\lambda^2 R_\lambda - \lambda I)(\mu^2 R_\mu - \mu I)f$, and these two factors commute because R_λ and R_μ commute

(iii) $\|A_\lambda - Af\| = \|\lambda R_\lambda Af - Af\| \xrightarrow{\lambda\to\infty} 0$ by Lemma 1.3.4 (ii)

□

Theorem 1.4.3. *(Hille-Yosida) A linear operator $L : D(L) \subset B \to B$ is the infinitesimal generator of a strongly continuous contractible semigroup $\{P_t\}_{t\geq 0}$ on B if and only if:*

(i) $\overline{D(L)} = B (= B_0)$ *and*

(ii) $(0, \infty) \subset \rho(L)$ *with* $\|\lambda R_\lambda\| \leq 1$ *for all* $\lambda \in \rho(L)$

Proof. ($\Rightarrow$) By Lemma 1.3.4, $D(L) = R_\lambda B$, and by Theorem 1.3.6, $\overline{R_\lambda B} = B_0$ ($= B$ by strong continuity), so statement (i) is established. Statement (ii) follows from Theorem 1.3.6 (i) in particular because

$$\|\lambda R_\lambda f\| \leq \lambda \int_0^\infty e^{-\lambda t}\|P_t f\| dt \leq \|f\|.$$

($\Leftarrow$) Let P_t^λ be the semigroup generated by L_λ, the Yosida approximation of L ($P_t^\lambda := e^{tL_\lambda}$ would be a reasonable construction). *Claim*: $\left\{P_t^\lambda\right\}_\lambda$ is Cauchy with respect to λ locally on compacts.

Proof of claim: Let $\lambda, \mu \in \mathbb{R}$. Then for $f \in B$,

$$\begin{aligned}\left\|P_t^\lambda f - P_t^\mu f\right\| &= \left\|P_t^\lambda P_0^\mu f - P_0^\lambda P_t^\mu f\right\| \\ &= \left\|\sum_{k=1}^n P_{tk/n}^\lambda P_{t(n-k)/n}^\mu f - P_{t(k-1)/n}^\lambda P_{t(n-(k-1))/n}^\mu f\right\| \\ &\leq \sum_{k=1}^n \left\|P_{t(k-1)/n}^\lambda P_{t(n-k)/n}^\mu \left(P_{t/n}^\lambda f - P_{t/n}^\mu f\right)\right\|\end{aligned}$$

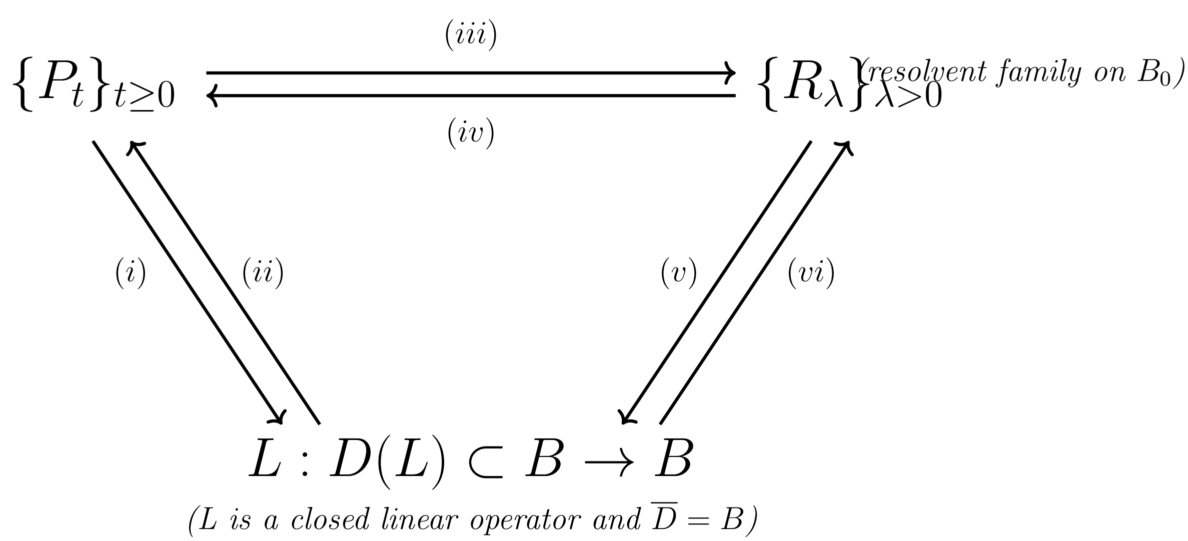
$$\begin{aligned}
&\leq \sum_{k=1}^{n} \left\| P^\lambda_{t(k-1)/n} \right\| \left\| P^\mu_{t(n-k)/n} \right\| \left\| P^\lambda_{t/n} f - P^\mu_{t/n} f \right\| \leq n \left\| P^\lambda_{t/n} f - P^\mu_{t/n} f \right\| \\
&\leq t \left\| \frac{n}{t} \left(P^\lambda_{t/n} f - f \right) - L_\lambda f \right\| + t \left\| \frac{n}{t} \left(P^\mu_{t/n} f - f \right) - L_\mu f \right\| + t \left\| L_\lambda f - L_\mu f \right\| \\
&\xrightarrow{n\to\infty} t \left\| L_\lambda f - L_\mu f \right\| \xrightarrow{\lambda,\mu\to\infty} 0
\end{aligned}$$

for any t, so the claim is proven on $D(L)$. This can be extended to all of B by the density of $D(L)$ and the claim is proven.
Hence, there exists $P_t f := \underset{\lambda\to\infty}{\text{s-lim}}\, P^\lambda_t f$ with uniform convergence on compact subsets. Now it remains to check that $\{P_t\}_{t\geq 0}$ as defined above is indeed a strongly continuous contractive semigroup and that L is indeed the infinitesimal generator of $\{P_t\}_{t\geq 0}$. The former is left as a homework exercise.
For the latter, let H be the infinitesimal generator of $\{P_t\}_{t\geq 0}$. Then for $f \in D(L)$, we have

$$\left\| \frac{1}{t} (P_t f - f) - Lf \right\| \leq \left\| \frac{1}{t} \left(P_t f - P^\lambda_t f \right) \right\| + \left\| \frac{1}{t} \left(P^\lambda_t f - f \right) - L_\lambda f \right\| + \left\| L_\lambda f - Lf \right\| \xrightarrow{\lambda\to\infty} 0.$$

Showing the first of these terms tends to 0 is left as a homework exercise, but it is very similar to the above "telescopic series" argument. Also, note that $D(L) = R^L_1 B = R^H_1 B = D(H)$, since the semigroups generated by L and H are the same, meaning their resolvents are the same. □

Proposition 1.4.4. *(Diagram)*



Proof. (i) $Lf = s - \lim_{t\to 0} \frac{1}{t}(P_t f - f)$, see Definition 1.2.2.

(ii) $P_t = e^{tL}$

(iii) Using the Laplace Transform (1.3.6), $R_\lambda f = \int_0^\infty e^{-\lambda s} P_s f ds$

(iv) Hille-Yosida

(v) $L = \lambda I - R^{-1}$

(vi) $R_\lambda = (\lambda I - L)^{-1}$

□

Definition 1.4.5. A semigroup $\{P_t\}_{t\geq 0}$ is *conservative* if

$$P_t 1 = 1 \ \forall t \geq 0.$$

Theorem 1.4.6. *Given the compact metric space (M, d), a closed linear operator $L : D(L) \subset C(M) \to C(M)$ is the infinitesimal generator of a conservative strongly continuous contractive semigroup, $\{P_t\}_{t\geq 0}$ if and only if:*

(i) $\overline{D(L)} = C(M)$ $(= C_0(M) = C_b(M))$

(ii) $\forall \lambda > 0$ *and* $g \in C(M)$ $\exists$ *at least one* $f \in D(L)$ *such that* $\lambda f - Lf = g$

(iii) The maximum principle holds: $\forall f \in D(L)$ *and* $x_0 \in M$ *such that* $f(x_0) = \|f\|_\infty$*, it follows that* $Lf(x_0) \leq 0$

(iv) $1 \in D(L)$ *and* $L1 = 0$.

Proof. ($\Rightarrow$)

(i) Theorem (1.2.3) (i) $\Rightarrow$ $\overline{D(L)} = B_0 = C(M)$ (because M is compact)

(ii) Theorem (1.3.6) $\Rightarrow (0, \infty) \subseteq \rho(L) \Rightarrow g = R_\lambda^{-1} R_\lambda g = (\lambda I - L) R_\lambda g =: f \in D(L)$ ($D(L) = R_\lambda B$ by Lemma (1.3.4))

(iii) We want $Lf(x_0) \leq 0$ given $x_0 \in M$ and $f(x_0) = \|f\|_\infty$. We check that $(P_t f)(x_0) - f(x_0) \leq 0$:

$$P_t f(x_0) \leq \|P_t f\|_\infty \leq \|f\|_\infty = f(x_0)$$

(the second inequality holds by contractibility)

(iv) $L1 = \text{s-}\lim_{t\to 0} \frac{1}{t}(P_t 1 - 1) = 0$ ($P_t 1 = 1$).

($\Leftarrow$) The result(s) will follow by *Hille-Yosida* if we verify the necessary conditions:

1. Check that $(0, \infty) \subseteq \rho(L)$

   Let $\lambda > 0$. We want to show that $(\lambda I - L)f = 0 \Rightarrow f = 0$. Suppose $f \neq 0$. Then $\exists x_0 \in M$ such that $\frac{1}{\lambda} Lf(x_0) = f(x_0) = \|f\|_\infty > 0 (WLOG) \Rightarrow Lf(x_0) > 0$

2. Check that $\|\lambda R_\lambda\|_\infty \leq 1 \ \forall \lambda$

   By compactness of M, for some x_0, $\lambda f(x_0) = \|\lambda f\|_\infty = \left\|\lambda R_\lambda R_\lambda^{-1} f\right\|_\infty = \|\lambda R_\lambda g\|_\infty$ with $g = R_\lambda^{-1} f$.
   At the same time

   $$\begin{aligned} \lambda f(x_0) &= \lambda f(x_0) - Lf(x_0) + Lf(x_0) = (\lambda I - L) f(x_0) + Lf(x_0) \\ &= R_\lambda^{-1} f(x_0) + Lf(x_0) = g(x_0) + Lf(x_0) \\ &\leq \|\lambda g\|_\infty \end{aligned}$$

   ($Lf(x_0) \leq 0$ by the maximal principle)

It follows that R_λ is bounded and L is closed. *Hille-Yosida* $\Rightarrow$ The exists a semigroup $\{P_t\}_{t \geq 0}$ with L as the infinitesimal generator. Moreover, the semigroup is is conservative due to the following:

$$P_t 1 = \sum_{k=0}^{\infty} \frac{t^k}{k!} L^k 1 = 1$$

($L^k 1 = 0$ for $k \geq 1$). □

Example 1.4.7. Say $M = [0, 1]$ $Lf = \frac{1}{2} f''$ and $D(L) = \{f \in C^2(M) : \ f'(0) = f'(1) = 0\}$. Check the conditions:

(i) $\overline{D(L)} = C([0, 1])$ by *Stone-Weierstrass.*

(ii) $\forall g \in C(M)$, f is the solution to

$$\begin{aligned} \lambda f - \frac{1}{2} f'' = g \\ f'(0) = f'(1) = 0 \end{aligned}$$

(To solve for f is left as a homework exercise)

(iii) maximum principle: Show $f(x_0) = \|f\|_\infty \Rightarrow f''(x_0) \leq 0$

For $x_0 \in (0, 1)$ the maximum principle is satisfied for $f \in C^2(M)$
For $x_0 \in \{0, 1\}$,

(iv) The last property follows from the setup

1.5. Stochastic processes and semigroups

Lemma 1.5.1. *Let $X = \{X_t\}_{t\geq 0}$ be a $\mathcal{F}_t$ adapted process. Then, TFAE:*

(i) (Basic Markov Property) $\mathbb{P}(X_t \in A \mid \mathcal{F}_s) = \mathbb{P}(X_t \in A \mid \sigma(X_s))$, $\forall A \in \mathcal{B}(M), 0 \leq s < t \leq \infty$.

(ii) $\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[f(X_t) \mid \sigma(X_s)]$, $\forall f \in B, 0 \leq s \leq t$.

(iii) If $s \geq 0$ *and* $E \in \sigma(X_t \ : \ t \geq s)$, *then* $\mathbb{P}(E \mid \mathcal{F}_s) = \mathbb{P}(E \mid \sigma(X_s))$.

(iv) $\forall s \geq 0$, $\mathcal{F} \in \mathcal{F}_s$ *and* $E \in \sigma(X_t \ : \ t \geq s)$, $\mathbb{P}(E \cap F \mid \sigma(X_s)) = \mathbb{P}(E \mid \sigma(X_s)) \cdot \mathbb{P}(F \mid \sigma(X_s))$.

Proof. In order to show that (ii) $\implies$ (i), we assume $f = \mathbf{1}_A$ in (ii). Now, to show that (i) $\implies$ (ii), we first note that (ii) follows immediately for characteristic functions $(\mathbf{1}_A, A \in \mathcal{B}(M))$. Then, we follow the process of *measure-theoretic induction* (prove it for simple functions, non-negative measurable and any measurable) to show that it follows $\forall f \in B$.
We now show that (iii) $\implies$ (iv). Consider $\mathcal{F} \in \mathcal{F}_s$ and $E \in \sigma(X_t \ : \ t \geq s)$. Then,

$$\mathbb{P}(E \cap F \mid \sigma(X_s)) = \mathbb{E}[\mathbf{1}_E \cdot \mathbf{1}_F \mid \sigma(X_s)]$$

Note that $\sigma(X_s) \subset \mathcal{F}_s$ and use the tower property

$$= \mathbb{E}[\mathbb{E}[\mathbf{1}_E \cdot \mathbf{1}_F \mid \sigma(X_s)] \mid \mathcal{F}_s]$$
$$= \mathbb{E}[\mathbf{1}_F \ \mathbb{E}[\mathbf{1}_E \mid \mathcal{F}_s] \mid \sigma(X_s)]$$

Since $\sigma(X_s)$ is measurable,

$$= \mathbb{E}[\mathbf{1}_F \mathbb{P}[E \mid \mathcal{F}_s] | \sigma(X_s)]$$
$$= \mathbb{P}(E \, | \mathcal{F}_s) \mathbb{E}[\mathbf{1}_F \mid \sigma(X_s)] = \mathbb{P}(E \, | \mathcal{F}_s) \cdot \mathbb{P}(F \, | \mathcal{F}_s).$$

□

Definition 1.5.2. Let $\{\mathbb{P}_x\}_{x\in M}$ be a family of probability measures and $\{\mathcal{F}_t\}_{t\geq 0}$ a filtrartion for $(\Omega, \mathcal{F})$. A stochastic process $X = \{X_t\}_{t>0}$ is Markov if

(i) X is $\mathcal{F}_t$ adapted.

(ii) for each $E \subset \mathcal{F}_\infty = \sigma\Big(\bigcup\limits_{t\geq 0} \mathcal{F}_t\Big)$, the map $x \mapsto \mathbb{P}_x(E)$ is $\mathcal{B}(M)$-measurable.

(iii) the (weak) Markov property holds, i.e. $\forall x \in M, A \in \mathcal{B}(M), s, t \geq 0$,

$$\mathbb{P}_x(X_{t+s} \in A \mid \mathcal{F}_s) = \mathbb{P}_{X_s}(X_t \in A).$$

Proposition 1.5.3. *A Markov process* $X = \{X_t\}_{t\geq 0}$ *satisfies the basic Markov property for any* $\mathbb{P}_x$, $x \in M$.

Proof. Consider a Markov process $X = \{X_t\}_{t\geq 0}$ and a $x \in M, \mathbb{P}_x$

$$\mathbb{P}_x(X_t \in A \mid \mathcal{F}_s) = \mathbb{P}_x(X_{t-s+s} \in A \mid \mathcal{F}_s)$$

by the (weak) Markov property,

$$\begin{aligned} &= \mathbb{P}_{X_s}(X_{t-s} \in A) \\ &= \mathbb{E}_x[\mathbb{P}_{X_s}(X_{t-s} \in A \mid \sigma(X_s)] \end{aligned}$$

by the (weak) Markov property,

$$= \mathbb{E}_x[\mathbb{P}_x(X_t \in A \mid \mathcal{F}_s) \mid \sigma(X_s)]$$

Note that $\sigma(X_s) \subset \mathcal{F}_s$ and $\mathbb{P}_x(X_t \in A \mid \mathcal{F}_s) = \mathbb{E}_x[\mathbf{1}_{\{X_t\in A\}} \mid \mathcal{F}_s]$. Using the tower property,

$$= \mathbb{E}[\mathbf{1}_{\{X_t\in A\}} \mid \sigma(X_s)] = \mathbb{P}_x(X_t \in A \mid \sigma(X_s)).$$

□

Definition 1.5.4. A function $p : [0,\infty) \times M \times \mathcal{B}(M) \to [0,1]$ is called a time-homogeneous Markov transition function if

(i) $\forall t \geq 0$ and $x \in M$, $A \mapsto p(t,x,A)$ is a measure on M.

(ii) $\forall t \geq 0$ and $A \in \mathcal{B}(M)$, $x \mapsto p(t,x,A)$ is $\mathcal{B}(M)$-measurable.

(iii) (Chapman-Kolmogorov Property) $\forall s,t \geq 0$, $x \in M, A \in \mathcal{B}(M)$

$$p(t+s,x,A) = p(t,\cdot,\cdot) * p(s,\cdot,\cdot)(x,A) = \int_M p(s,y,A)p(t,x,dy).$$

Note: (i) and (ii) are also known as the transition kernel and $\int_M p(s,y,A)p(t,x,dy)$ can also be represented as $\int_M p(s,y,A)dp(t,x,y)$.

Example 1.5.5.

$$p^{BM}(t,x,A) = \begin{cases} \mathbf{1}_A(x), & \text{t=0} \\ \int_A \frac{1}{\sqrt{2\pi t}} e^{\frac{-|x-y|^2}{2t}} dy & t > 0 \end{cases}$$

is a Markov transition function

Definition 1.5.6. Let $X = \{X_t\}$ be a $\mathcal{F}_t$-adapted Markov process. A Markov transition function $p : [0,\infty) \times M \times \mathcal{B}(M) \to [0,1]$ is a transitions function for X if for all $s,t \geq 0$ and $A \in \mathcal{B}(M)$

$$p(t,X_s,A) = \mathbb{P}_x(X_{t+s} \in A \mid \mathcal{F}_s) \quad \mathbb{P}_x - \text{a.s.}$$

Theorem 1.5.7. *Consider a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and a probability measure $\{\mathbb{P}_x\}_{x\in M}$ Let $X = \{X_t\}_{t\geq 0}$ be a Markov process. Then,*

(i) for $t \geq 0, x \in M, A \in \mathcal{B}(M)$

$$p(t, x, A) = \mathbb{P}_x(X_t \in A).$$

(ii) the family of operators $P_t : B \to B$ s.t. $f \mapsto \int_M f(y)p(t, x, dy) = \mathbb{E}_x[f(X_t)]$ is a contraction semigroup on B.

Proof. (i) $A \mapsto p(t, x, A)$ defines a measure by definition of $x \mapsto p(t, x, A)$ is a $\mathcal{B}(M)$-measurable because X is Markov.
Now, we will show that Chapman-Kolmogorov property holds.

$$\begin{aligned} p(t+s, x, A) &= \mathbb{P}_x(X_{s+t} \in A) \\ &= \mathbb{E}_x[\mathbb{E}_x[\mathbf{1}_{\{X_t\in A\}} \mid \mathcal{F}_s]] = \mathbb{E}_x[\mathbb{P}_x(X_t \in A \mid \mathcal{F}_s)] \end{aligned}$$

by the (weak) Markov property

$$= \mathbb{E}_x[\mathbb{P}_{X_s}(X_t \in A)] = \int_\Omega p(t, X_s(\omega), A)d\mathbb{P}_x(\omega)$$

set $y = X_s(\omega)$

$$\begin{aligned} &= \int_M p(t, y, A)d\mathbb{P}_x \circ X_s^{-1}(y) = \int_M p(t, y, A)\mathbb{P}_x(X_s \in dy) \\ &= \int_M p(t, y, A)p(s, x, dy). \end{aligned}$$

(ii) We will first show that P_t is a semigroup.

$$P_0 f(x) = \int_M f(y)p(0, x, dy) = \int_M f(y)\mathbb{P}_x(X_0 \in dy) = f(x).$$

Also,

$$\begin{aligned} P_{t+s}f(x) &= \int_M f(z)p(t+s, x, dz) \\ &= \int_M \left[\int_M f(z)p(t, y, dz)\right] p(s, x, dy) \end{aligned}$$

$$= \int_M P_t f(y) p(s,x,dy) = P_s(P_t f)(x).$$

Now, to show that P_t is contractive, we note that for all $x \in M$,

$$|P_t f(x)| \leq \|f\| \int_M p(t,x,dy) \leq \|f\|$$

therefore $\|P_t f\| \leq \|f\|$.

□

Proposition 1.5.8. *Let $X = \{X_t\}_{t\geq 0}$ be a Markov process. Assume:*

(i) The Markov transition function is symmetric i.e. $\forall f, g \in B$ with compact support,

$$\int_M \int_M f(x)g(y)p(t,x,dy)\mu(dx) = \int_M \int_M f(y)g(x)p(t,x,dy)\mu(dx).$$

(ii) $\exists D \subset B \cap L'(M,\mu)$ dense in $L^2(M,\mu)$ such that $\forall f \in D$,

$$\lim_{t\to 0^+} \int_M f(y)p(t,x,dy) = f(x) \quad \mu - a.e. \text{ for } x \in M.$$

Then, the associated semigroup $P_t f(x) = \int_M f(y)p(t,x,dy)$ in $L^2(M,\mu)$ is a strongly contractive semigroup.

Proof. We only need to show that the semigroup is strongly contractive i.e. for any $f \in L^2(M,\mu)$, $\lim_{t\to 0} P_t f = f$ or $\lim_{t\to 0} \|P_t f - f\|_{L^2} = 0$. Note that D is dense in $L^2(M,\mu)$, so it is sufficient to prove the previous limits holds for any $f \in D$. So, consider a $f \in D$,

$$\|P_t f - f\|_{L^2}^2 = \int_M (P_t f - f)^2 d\mu = \int_M P_t f^2 d\mu - 2\int_M P_t f \cdot f d\mu + \int_M f^2 d\mu.$$

By the dominated convergence theorem

$$\lim_{t\to 0} \int_M P_t f \cdot f d\mu = \int_M \lim_{t\to 0} P_t f \cdot f d\mu = \int_M f^2 d\mu.$$

Moreover,

$$\int_M P_t f^2 d\mu = \int_M \Big(\int_M f(y)p(t,x,dy)\Big)^2 \mu(dx)$$

Using Cauchy-Schwarz

$$\leq \int_M \int_M f^2(y)p(t,x,dy) \int_M p(t,x,dy)\mu(dx)$$

Since $p(t,x,dy) \leq 1$

$$\leq \int_M \int_M f^2(y)p(t,x,dy)\mu(dx)$$

Using symmetry,

$$= \int_M \int_M f^2(x)p(t,x,dy)\mu(dx)$$
$$= \int_M f^2(x) \int_M p(t,x,dy)\mu(dx) \leq \int_M f^2 d\mu$$

□

Therefore,

$$\begin{aligned}
\lim_{t\to 0} \|P_t f - f\|^2_{L^2} &= \lim_{t\to 0} \int_M (P_t f - f)^2 d\mu \\
&= \lim_{t\to 0} \int_M P_t f^2 d\mu - 2 \lim_{t\to 0} \int_M P_t f \cdot f d\mu + \int_M f^2 d\mu \\
&= \lim_{t\to 0} \int_M P_t f^2 d\mu - 2 \int_M f^2 d\mu + \int_M f^2 d\mu \\
&= \lim_{t\to 0} \int_M P_t f^2 d\mu - \int_M f^2 d\mu = 0
\end{aligned}$$

Theorem 1.5.9. *Let $p : [0,\infty) \times M \times \mathcal{B}(M) \to [0,1]$ be a Markov transition function and μ a probability distribution on (M,d). There exists a Markov process $X = \{X_t\}_{t\geq 0}$ whose finite-dimensional distributions are uniquely determines by*

$$\mathbb{P}_x(X_0 \in A_0, X_{t_1} \in A_1, ..., X_{t_n} \in A_n) = \int_{A_0} \int_{A_1} ... \int_{A_n} p(t_n - t_{n-1}, y_{n-1}, A_n)...p(t, y_0, dy)\mu(dy_0).$$

Proof. This follows from Kolmogorov's theorem using the fact that the family of measures $\left\{ \int_M p(t,x,\cdot)\mu(dx) \right\}_{t\geq 0}$ is tight. □

CHAPTER 2

Dirichlet form, infinitesimal generators, Markov semigroups

In this chapter, $\mathcal{H}$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\| \cdot \|_{\mathcal{H}}$.

2.1. Closed forms and generators

Definition 2.1.1. A densely defined non-negative symmetric bilinear form $\mathcal{E}$: $D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ (where $D(\mathcal{E}) \subset \mathcal{H}$) satisfies

(i) $D(\mathcal{E})$ is a dense linear subspace of $\mathcal{H}$.

(ii) $\forall f, g \in D(\mathcal{E})$, $\mathcal{E}(f, g) \geq 0$.

(iii) $\forall f, g \in D(\mathcal{E})$, $\mathcal{E}(f, g) = \mathcal{E}(g, f)$

(iv) $\forall f, g, h \in D(\mathcal{E})$, $\mathcal{E}(af + bg, h) = a\mathcal{E}(f, h) + b\mathcal{E}(g, h)$.

Then, $(\mathcal{E}, D(\mathcal{E}))$ is called a symmetric form.

Definition 2.1.2. A symmetric bilinear form $(\mathcal{E}, D(\mathcal{E}))$ is closed if $D(\mathcal{E})$ is a Hilbert space (i.e. complete) with the norm

$$\|f\|_{\mathcal{E}_1}^2 := \mathcal{E}(f, f) + \|f\|_{\mathcal{H}}^2, \quad \forall f \in D(\mathcal{E}).$$

Proposition 2.1.3. *For any $\lambda > 0$, let $\| \cdot \|_{\mathcal{E}_\lambda}$ be the norm $\|f\|_{\mathcal{E}_\lambda} = \mathcal{E}(f, f) + \lambda \|f\|_{\mathcal{H}}^2$. For any $\lambda_1, \lambda_2 > 0$, $\| \cdot \|_{\mathcal{E}_{\lambda_1}}$ is comparable to $\| \cdot \|_{\mathcal{E}_{\lambda_2}}$.*

Proof.

$$\|f\|_{\mathcal{E}_{\lambda_1}} = \mathcal{E}(f, f) + \lambda_1 \|f\|_{\mathcal{H}}^2 \leq \|f\|_{\mathcal{E}_{\lambda_2}} + \frac{\lambda_1}{\lambda_2} \|f\|_{\mathcal{E}_{\lambda_2}} = (1 + \frac{\lambda_1}{\lambda_2}) \|f\|_{\mathcal{E}_{\lambda_2}}.$$

□

Definition 2.1.4. A symmetric form $\mathcal{E}$ is closable if for all sequence $f_n \in D(\mathcal{E})$,

$$\left.\begin{aligned} \mathcal{E}(f_n - f_m, f_n - f_m) &\xrightarrow{n,m\to\infty} 0 \\ \|f_n\|_{\mathcal{H}} &\xrightarrow{n\to\infty} 0 \end{aligned}\right\} \implies \mathcal{E}(f_n, f_n) \xrightarrow{n\to\infty} 0.$$

Theorem 2.1.5. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a closed symmetric form on $\mathcal{H}$. There is a unique non-positive self-adjoint operator $L : D(L) \subseteq \mathcal{H} \to \mathcal{H}$ such that*

(i) $D(L) = \{f \in D(\mathcal{E}) : \exists h \in \mathcal{H} \text{ s.t. } \mathcal{E}(f,g) = -\langle h, g\rangle_{\mathcal{H}}, \forall g \in D(\mathcal{E})\}$. *And we set* $Lf = h$.

(ii) $\overline{D(L)}^{\mathcal{E}_1} = D(\mathcal{E})$.

Proof. (i) For any $\lambda > 0$, since $(D(\mathcal{E}, \|\cdot\|_{\mathcal{E}_\lambda})$ is a Hilbert space, for all $f \in \mathcal{H}$,

$$\exists! h \in D(\mathcal{E}), \text{ s.t. } \langle h, g\rangle_{\mathcal{E}_\lambda} = \langle f, g\rangle, \; \|h\|_{\mathcal{E}_\lambda} \le \|f\|/\sqrt{\lambda}, \quad \forall g \in D(\mathcal{E}).$$

Denote $h = \tilde{R}_\lambda f$, then $\tilde{R}_\lambda$ is a bounded operator on both $\mathcal{H}$ and $(D(\mathcal{E}), \|\cdot\|_{\mathcal{E}_\lambda})$ with $\|\tilde{R}_\lambda\| \le 1/\lambda$ and $\|\tilde{R}_\lambda\|_{D(\mathcal{E}),\mathcal{E}_\lambda} \le 1/\lambda$. And we have

$$\mathcal{E}(\tilde{R}_\lambda f, g) + \lambda\langle \tilde{R}_\lambda f, g\rangle = \langle f, g\rangle \tag{2.1.1}$$

We want to show that $\tilde{R}_\lambda$ is actually the resolvent of L so that $Lf = (\lambda - \tilde{R}_\lambda^{-1})f$. First, we check some basic properties of $\tilde{R}_\lambda$.

(a) $\tilde{R}_{\lambda_1} - \tilde{R}_{\lambda_2} = (\lambda_2 - \lambda_1)\tilde{R}_{\lambda_2}\tilde{R}_{\lambda_1}$. Indeed, substituting λ_1,λ_2 into (2.1.1) and taking the difference, we obtain

$$\mathcal{E}(\tilde{R}_{\lambda_1} f - \tilde{R}_{\lambda_2} f, g) + \lambda_2\langle \tilde{R}_{\lambda_1} f - \tilde{R}_{\lambda_2} f, g\rangle = \langle (\lambda_2 - \lambda_1)\tilde{R}_{\lambda_1} f, g\rangle.$$

Applying (2.1.1) again to the above equation, we have

$$\tilde{R}_{\lambda_1} f - \tilde{R}_{\lambda_2} f = (\lambda_2 - \lambda_1)\tilde{R}_{\lambda_1}\tilde{R}_{\lambda_2} f.$$

(b) $f = \lim_{\lambda\to\infty} \lambda\tilde{R}_\lambda f$ in $\mathcal{H}$ for all $f \in D(\mathcal{E})$. See the proof of Lemma 1.3.4 (iii).

(c) $\tilde{R}_\lambda^{-1}$ exists: $\tilde{R}_\lambda f = 0 \implies f = 0$. Assume $\tilde{R}_\lambda f = 0$. Fix $\lambda > 0$, for any $\tilde{\lambda} > \lambda$, by (a)

$$\tilde{R}_{\tilde{\lambda}} f = \tilde{R}_\lambda f + (\lambda - \tilde{\lambda})\tilde{R}_{\tilde{\lambda}}\tilde{R}_\lambda f = 0.$$

Therefore $f = \lim_{\lambda\to\infty} \lambda\tilde{R}_\lambda f = 0$.

Now define L as $Lf = (\lambda - \tilde{R}_\lambda^{-1})f$ with $D(L) = \tilde{R}_\lambda \mathcal{H}$. Using (a) one can easily check that this definition is independent of λ. In particular, $\{\tilde{R}_\lambda\}_{\lambda>0}$ is the resolvent of L. So $(0,\infty) \subset \rho(L)$ hence L is closed (also densely defined by (b)).
We now check that L indeed satisfies (i). For any $f = \tilde{R}_\lambda k \in \tilde{R}_\lambda \mathcal{H}$, by (2.1.1), $\mathcal{E}(f,g) = -\langle \lambda f - k, g\rangle$. So

$$\tilde{R}_\lambda \mathcal{H} \subseteq \{f \in D(\mathcal{E}) : \exists h \in \mathcal{H} \text{ s.t. } \mathcal{E}(f,g) = -\langle h, g\rangle_{\mathcal{H}}, \forall g \in D(\mathcal{E})\}.$$

Conversely, if for $f \in D(\mathcal{E})$ there is $h \in \mathcal{H}$ s.t. $\mathcal{E}(f,g) = -\langle h,g\rangle_{\mathcal{H}}$ for all $g \in D(\mathcal{E})$, then we have

$$-\mathcal{E}(f,g) + \lambda\langle f,g\rangle = \langle \lambda f - h, g\rangle.$$

So $f = \tilde{R}_\lambda(\lambda f - h) \in \tilde{R}_\lambda \mathcal{H}$ by (2.1.1). Therefore,

$$D(L) = \tilde{R}_\lambda \mathcal{H} = \{f \in D(\mathcal{E}) : \exists h \in \mathcal{H} \text{ s.t. } \mathcal{E}(f,g) = -\langle h, g\rangle_{\mathcal{H}}, \forall g \in D(\mathcal{E})\}.$$

This implies that L indeed satisfies (i).
(ii) We can now again proceed as in the proof of Lemma 1.3.4 (iii) to show that $\overline{D(L)}^{\mathcal{E}_1} = D(\mathcal{E})$ using $\|\lambda \tilde{R}_\lambda\|_{D(\mathcal{E}),\mathcal{E}_\lambda} \leq 1$.
Finally, since $\mathcal{E}$ is symmetric, L is also symmetric ($D(L) \subset D(L^*)$, $L \subset L^*$). It remains to check that L is self-adjoint, i.e. $D(L^*) \subseteq D(L)$. If $f \in D(L^*)$, then for all $k \in D(L)$, $\langle L^* f, k\rangle = \langle f, Lk\rangle = \langle Lk, f\rangle = -\mathcal{E}(f,k)$. Since $\overline{D(L)}^{\mathcal{E}_1} = D(\mathcal{E})$, by continuity we have for all $k \in D(\mathcal{E})$, $\langle L^* f, k\rangle = -\mathcal{E}(f,k)$, hence $f \in D(L)$. □

Example 2.1.6. $\mathcal{E}(f,g) = \int_{\mathbb{R}} f' \cdot g' dx$, with $D(\mathcal{E}) = C_0^2(\mathbb{R})$ the space of compactly supported C^2 functions. Then $\mathcal{E}$ is closable and symmetric in $L^2(\mathbb{R}, dx)$.

$$\mathcal{E}(f,g) = \int_{\mathbb{R}} f' \cdot g' dx = -\int f'' \cdot g dx = -\langle f'', g\rangle.$$

Thus $L = \frac{d}{dx^2} \leq 0$, and $D(L) = \overline{C_c^2(\mathbb{R})}^{\mathcal{E}_1}$ where $\mathcal{E}_1$ is the Sobolev norm $\|f\|^2_{\mathcal{E}_1} = \|f\|^2_{L^2} + \|f'\|^2_{L^2}$.

Theorem 2.1.7 (Spectral Theorem)**.** *Let $L : D(L) \subseteq \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator with spectrum $\sigma(L)$. There is a finite measure ν on $\sigma(L) \times \mathbb{N}$ and a unitary operator $U : \mathcal{H} \to L^2(\sigma(\nu) \times \mathbb{N}, \nu)$ such that*

(i) $f \in D(L) \Leftrightarrow g(s) := s \cdot Uf(s,n) \in L^2(\sigma(\nu) \times \mathbb{N}, \nu)$.

(ii) $ULU^{-1}\varphi(s) = s\varphi(s), \quad \forall \varphi \in U(D(L))$.

(iii) $UF(L)U^{-1}\varphi(s) = F(s)\varphi(s), \quad \forall F \in C_c(\mathbb{R}), \varphi \in L^2(\sigma(\nu) \times \mathbb{N}, \nu)$.

Theorem 2.1.8. *Let $L : D(L) \subseteq \mathcal{H} \to \mathcal{H}$ be a non-positive self-adjoint operator. Then,*

$$\mathcal{E}(f,g) = \langle (-L)^{1/2} f, (-L)^{1/2} g\rangle_{\mathcal{H}}, \quad D(\mathcal{E}) = D((-L)^{1/2}).$$

Proof. We want to show $(\mathcal{E}, D(\mathcal{E}))$ is closed, i.e. if $\{f_n\} \subseteq D(\mathcal{E})$ is $\mathcal{E}_1$-Cauchy and $\|f_n\|_{\mathcal{H}} \to 0$, then $\mathcal{E}(f_n, f_n) \to 0$. Since $-L$ is non-negative and self-adjoint, we can apply spectrum theorem (or functional calculus) to get that $(-L)^{1/2}$ is also non-negative and selfadjoint (in particular closed). Therefore if $(-L)^{1/2} f_n \xrightarrow{n\to\infty} g$ then $g = (-L)^{1/2} 0 = 0$. □

Theorem 2.1.9. *Let $L : D(L) \subseteq \mathcal{H} \to \mathcal{H}$ be a non-positive self-adjoint operator with associated closed symmetric form $(\mathcal{E}, D(\mathcal{E}))$. Then L is the infinitesimal generator of a strongly continuous contractive semigroup $\{P_t\}_{t\geq 0}$ on $\mathcal{H}$ and*

$$\mathcal{E}(f,g) = \lim_{t\to 0} \frac{1}{t}\langle f - P_t f, g\rangle, \quad D(\mathcal{E}) = \{f \in \mathcal{H} : \mathcal{E}(f,f) < \infty\}.$$

Proof. (1) The resolvent of L satisfies Hille-Yosida (HW), hence the associated semigroup $\{P_t\}_{t\geq 0}$ exists. And also by Hille-Yosida $P_t f = e^{tL} f$ for all $f \in \mathcal{H}$. By spectral theorem, $UP_tU^{-1}\varphi(s) = e^{ts}\varphi(s)$ for all $\varphi \in L^2(\sigma(L) \times \mathbb{N}, \nu)$. Now, for $f \in \mathcal{H}$, let $\varphi = Uf$

$$\begin{aligned}\frac{1}{t}\langle f - P_t f, f\rangle_{\mathcal{H}} &= \frac{1}{t}\langle (I - P_t) f, f\rangle_{\mathcal{H}} = \frac{1}{t}\langle U(I-P_t)U^{-1}Uf, Uf\rangle_{L^2} \\ &= \frac{1}{t}\langle U(I-P_t)U^{-1}\varphi, \varphi\rangle_{L^2} = \int_{\sigma(L)\times\mathbb{N}} (I - e^{tL})\varphi(s)\varphi(s)d\nu(s,n)\end{aligned}$$

Taking $t \to 0$, we see that $\lim_{t\to 0} \frac{1}{t}\langle f - P_t f, f\rangle_{\mathcal{H}}$ exists iff $\int_{\sigma(L)\times\mathbb{N}} s\varphi(s)\varphi(s)d\nu(s,n)$ is finite. But

$$\begin{aligned}\int_{\sigma(L)\times\mathbb{N}} s\varphi(s)\varphi(s)d\nu(s,n) &= \langle \sqrt{-s}\varphi(s), \sqrt{-s}\varphi(s)\rangle_{L^2} \\ &= \langle (-L)^{1/2} f, (-L)^{1/2} f\rangle_{\mathcal{H}} - \mathcal{E}(f,f),\end{aligned}$$

which is finite iff $f \in D((-L)^{1/2}) = D(\mathcal{E})$. □

2.2. Dirichlet forms and Markov semigroups

Definition 2.2.1. A linear operator $A : D(A) \subseteq L^2(M,\mu) \to L^2(M,\mu)$ is called Markovian if for any $f \in D(A)$ with $0 \leq f \leq 1$ μ-a.e., we have $0 \leq Af \leq 1$ μ-a.e. $\{P_t\}_{t\geq 0}$ is Markovian if P_t is Markovian for all $t > 0$.

Definition 2.2.2. (i) A normal contraction is a function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(0) = 0$ and $|\psi(s) - \psi(t)| \leq |s - t|$ for all $s, t \in \mathbb{R}$.

(ii) For $f \in M \to \mathbb{R}$, a normal contraction of f is any function $\tilde{f} = \psi \circ f$ where ψ is a normal contraction. In particular $|\tilde{f}(x) - \tilde{f}(y)| \leq |f(x) - f(y)|$ and $|\tilde{f}(x)| \leq |f(x)|$.

Definition 2.2.3. A closed symmetric form $(\mathcal{E}, D(\mathcal{E}))$ is said to be Markovian if for all $f \in D(\mathcal{E})$, any normal contraction $\tilde{f}$ of f satisfies

$$\tilde{f} \in D(\mathcal{E}) \text{ and } \mathcal{E}(\tilde{f}, \tilde{f}) \leq \mathcal{E}(f, f).$$

A Dirichlet form (D.F.) is a Markovian closed symmetric form.

Proposition 2.2.4. *A closed symmetric form $(\mathcal{E}, D(\mathcal{E}))$ is Markovian if for all $f \in D(\mathcal{E})$, we have*

$$0 \vee f \wedge 1 \in D(\mathcal{E}) \quad \textit{and} \quad \mathcal{E}(0 \vee f \wedge 1, 0 \vee f \wedge 1) \leq \mathcal{E}(f, f).$$

Theorem 2.2.5. *Let $\{P_t\}_{t\geq 0}$ be a (symmetric) strongly continuous semigroup in $L^2(M, \mu)$ that is Markovian. There exists a symmetric Markov transition function $p(t, x, \cdot)$ such that*

$$P_t f(x) = \int_M f(y) p(t, x, dy)$$

for all $f \in L^\infty(M, \mu)$.

The proof of this theorem uses the bi-measure theorem.

Theorem 2.2.6. *A closed symmetric form $(\mathcal{E}, D(\mathcal{E}))$ is Markovian iff the associated semigroup $\{P_t\}_{t\geq 0}$ is Markovian.*

We prove this next, but first we introduce the following definition.

Definition 2.2.7. Let H be a Hilbert space. We call $C \subset H$ a cone if for any $f \in C$, we have $\alpha f \in C$ for all $\alpha > 0$.

Lemma 2.2.8. *Let $C \subset H$ be a cone and assume that for all $f \in H$, there is $\tilde{f} \in H$ such that*

$$\|\tilde{f}\|_H \leq \|f\|_H \ (*) \quad \textit{and} \quad |\langle f, h\rangle_H| \leq \langle \tilde{f}, h\rangle_H \ (**)$$

for all $h \in C$. Then $f = \tilde{f}$ whenever $f \in C$.

Proof. Simply note that if $f \in C$, by using $(*)$ and $(**)$, we get

$$\|f - \tilde{f}\|_H^2 = \|f\|_H^2 - 2\langle f, \tilde{f}\rangle_H + \|\tilde{f}\|_H^2 \leq 0.$$

□

We now prove Theorem 2.2.6.

Proof of Theorem 2.2.6. We first prove the forward implication. We want to show that whenever $0 \leq f \leq 1$, we have $0 \leq P_t f \leq 1$. Recall that $\mathcal{E}_\lambda(f, g) = \mathcal{E}(f, g) + \lambda\langle f, g\rangle_{L^2}$ is an inner product in $D(\mathcal{E})$. Consider the cone $C = \{R_\lambda f : f \geq 0\}$. Let $g \in D(\mathcal{E})$. Note that

$$\mathcal{E}_\lambda(|g|, R_\lambda f) = \mathcal{E}(|g|, R_\lambda f) + \lambda\langle |g|, R_\lambda f\rangle_{L^2}$$

$$\begin{aligned}
&= -\langle |g|, LR_\lambda f\rangle_{L^2} + \lambda\langle |g|, R_\lambda f\rangle_{L^2}\\
&= \langle |g|, (\lambda I - L)R_\lambda f\rangle_{L^2}\\
&= \langle |g|, f\rangle_{L^2} \qquad (2.2.1)\\
&\geq |\langle g, f\rangle_{L^2}|\\
&= |\mathcal{E}_\lambda(g, R_\lambda f)|
\end{aligned}$$

where the second equality follows from theorem 2.1.5. Also note that

$$\big\||g|\big\|^2_{\mathcal{E}_\lambda} = \mathcal{E}(|g|,|g|) + \lambda\langle |g|,|g|\rangle_{L^2} \leq \mathcal{E}(g,g) + \lambda\|g\|^2_{L^2} = \mathcal{E}_\lambda(g,g)$$

since $|g|$ is a normal contraction of g and $(\mathcal{E}, D(\mathcal{E}))$ is Markovian. Then, by the previous lemma, $|R_\lambda f| = R_\lambda f$ and so $R_\lambda f \geq 0$. In particular, for $\lambda > 0$, we have $\lambda R_\lambda f = 0 \vee \lambda R_\lambda f$. Note that

$$g := (0 \vee \lambda R_\lambda f) \wedge 1$$

is a normal contraction of $\lambda R_\lambda f$, and so $\mathcal{E}(g,g) \leq \mathcal{E}(\lambda R_\lambda f, \lambda R_\lambda f)$. Thus

$$\begin{aligned}
\|\lambda R_\lambda f - g\|^2_{\mathcal{E}_\lambda} &= \lambda^2\mathcal{E}_\lambda(R_\lambda f, R_\lambda f) - 2\lambda\mathcal{E}_\lambda(R_\lambda f, g) + \mathcal{E}_\lambda(g,g)\\
&= \lambda^2\langle R_\lambda f, f\rangle_{L^2} - 2\lambda\langle f, g\rangle_{L^2} + \mathcal{E}(g,g) + \lambda\langle g, g\rangle\\
&\leq \lambda^2\langle R_\lambda f, f\rangle_{L^2} - 2\lambda\langle f, g\rangle_{L^2} + \mathcal{E}(\lambda R_\lambda f, \lambda R_\lambda f) + \lambda\langle g, g\rangle\\
&= \lambda^2\langle R_\lambda f, f\rangle_{L^2} + \lambda\|f-g\|^2_{L^2} + \lambda^2\mathcal{E}(R_\lambda f, R_\lambda f) - \lambda\langle f, f\rangle_{L^2} = 0,
\end{aligned}$$

where the second equality follows from (2.2.1), and we leave the last equality as an exercise, cf. Exercise 2.2.10. Thus $g = \lambda R_\lambda f$, whence $0 \leq \lambda R_\lambda f \leq 1$. By Hille-Yosida's approximation,

$$P_t f = \text{s-lim}_{\lambda\to\infty} e^{t\lambda(\lambda R_\lambda - I)} f = \text{s-}\lim_{\lambda\to\infty} e^{-t\lambda}\sum_{k=0}^{\infty}\frac{(t\lambda)^k}{k!}(\lambda R_\lambda)^k f,$$

so the above implies that $0 \leq P_t f \leq 1$, as desired. We now prove the reverse implication. We want to show that $\mathcal{E}(g,g) \leq \mathcal{E}(f,f)$ for any normal contraction g of f. Note that

$$\begin{aligned}
\langle g - P_t g, g\rangle_{L^2} &= \int_M g^2 d\mu - \int_M P_t g\cdot g d\mu\\
&= \int_M (1 - P_t 1)g^2 d\mu + \int_M (P_t 1\cdot g - P_t g)\cdot g d\mu\\
&= \int_M (1 - P_t 1)g^2 d\mu + \int_M\int_M (g^2(x) - g(x)g(y))p(t,x,dy)d\mu(x).
\end{aligned} \qquad (2.2.2)$$

Now, by the symmetric property of $p(t,x,dy)$, we have $\int_M g^2(x)p(t,x,dy) = \int_M g^2(y)p(t,x,dy)$, so by writing $g^2(x) = \frac{1}{2}g^2(x) + \frac{1}{2}g^2(x)$ and completing the square in the last expression gives

$$\begin{aligned}\langle g - P_t g, g\rangle_{L^2} &= \int_M (1-P_t1)g^2 d\mu + \frac{1}{2}\int_M\int_M (g(x)-g(y))^2 p(t,x,dy)d\mu(x)\\ &\leq \int_M (1-P_t1)f^2 d\mu + \frac{1}{2}\int_M\int_M (f(x)-f(y))^2 p(t,x,dy)d\mu(x)\\ &= \langle f - P_t f, f\rangle_{L^2}\end{aligned}$$

since g is a normal contraction of f. Therefore

$$\mathcal{E}(g,g) = \lim_{t\to 0}\frac{1}{t}\langle g - P_t g, g\rangle_{L_2} \leq \lim_{t\to 0}\frac{1}{t}\langle f - P_t f, f\rangle_{L_2} = \mathcal{E}(f,f).$$

□

Example 2.2.9. Let (V,E) be a finite graph and write $x \sim y$ whenever two vertices x,y are connected by an edge. Then

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{x\sim y}[f(x)-f(y)][g(x)-g(y)]$$

is a Dirichlet form on

$$D(\mathcal{E}) = \ell^2(V) = \big\{f : V \to \mathbb{R} : \sum_{x\in V} f^2(x)\mathrm{deg}(x) < \infty\big\},$$

with infinitesimal generator given by

$$Lf(x) = \frac{1}{\mathrm{deg}(x)}\sum_{y\sim x}(f(x)-f(y)).$$

If we denote the adjacency matrix of the graph by Adj and $q := \mathrm{deg}(x)$, we then have $L = \frac{1}{q}\mathrm{Adj} - \mathrm{Id}$ in matrix form.

Exercise 2.2.10. Let $(\mathcal{E}, D(\mathcal{E})$ be a Dirichlet form in $L^2(M,\mu)$. For any $\lambda > 0$, the bilinear form

$$\mathcal{E}^{(\lambda)}(f,g) := \langle \lambda(I - \lambda R_\lambda)f, g\rangle_{L^2}, \qquad f,g \in L^2(M,\mu)$$

is sometimes called the *Deny-Yosida* approximation of $(\mathcal{E}, D(\mathcal{E})$. Prove that

$$\mathcal{E}^{(\lambda)}(f,g) = \mathcal{E}(\lambda R_\lambda f, \lambda R_\lambda f) + \lambda\|f-g\|^2_{L^2}.$$

2.3. Energy measure & carré du champ operators

Proposition 2.3.1. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form in $L^2(M,\mu)$. For any non-negative $h \in L^\infty(M,\mu) \cap L^2(M,\mu)$ and $f \in D(\mathcal{E}) \cap L^\infty(M,\mu)$,*

$$0 \leq \mathcal{E}(f, hf) - \frac{1}{2}\mathcal{E}(h, f^2) \leq \|h\|_\infty \mathcal{E}(f,f).$$

Proof. To estimate $\mathcal{E}(f, hf)$ note that as before in (2.2.2),

$$\langle f - P_t f, hf\rangle_{L^2} = \int_M (1-P_t 1) f^2 h d\mu + \int_M\int_M (f^2(x) - f(x)f(y))p(t,x,dy)h(x)d\mu(x),$$

and to estimate $\frac{1}{2}\mathcal{E}(h, f^2)$,

$$\frac{1}{2}\langle h - P_t h, f^2\rangle_{L^2} = \frac{1}{2}\int_M (1-P_t 1) f^2 h d\mu + \frac{1}{2}\int_M\int_M (h(x)-h(y))p(t,x,dy)f^2(x)d\mu(x).$$

Subtracting the two gives

$$\begin{aligned}
& \frac{1}{2}\int_M (1-P_t 1) f^2 h d\mu + \int_M\int_M \Big[\frac{1}{2}f^2(x)h(x) - f(x)f(y)h(x) + \frac{1}{2}f^2(x)h(y)\Big]p(t,x,dy)d\mu(x) \\
= \; & \frac{1}{2}\int_M (1-P_t 1) f^2 h d\mu + \frac{1}{2}\int_M\int_M h(x)(f(x)-f(y))^2 p(t,x,dy)d\mu(x) \\
\leq \; & \|h\|_\infty \Big[\int_M (1-P_t 1) f^2 d\mu + \frac{1}{2}\int_M\int_M (f(x)-f(y))^2 p(t,x,dy)d\mu(x)\Big] \\
= \; & \|h\|_\infty \langle f - P_t f, f\rangle_{L^2},
\end{aligned}$$

where we use the symmetric property of $p(t,x,dy)$ to then complete the square in the first equality. The claim follows by dividing over t and taking the limit as $t \to 0$. □

Theorem 2.3.2. *For any $f \in D(\mathcal{E}) \cap C_c(M)$ there exists a unique Borel measure ν_f such that*

$$F_f(h) := \mathcal{E}(f, hf) - \frac{1}{2}\mathcal{E}(h, f^2) = \int_M h d\nu_f$$

for all $h \in C_c(M)$.

Proof. By the last proposition, the linear function $F_f(h)$ is non-negative and continuous, so by Riesz-Markov's representation theorem, we obtain the existence of the Borel measure ν_f. □

Recall that ν_f satisfies:

$$\forall K \text{ compact}, \; \nu_f(K) < \infty,$$

$$\forall E \in \mathcal{B}(M), \; \nu_f(E) = \inf\{\nu_f(U) : E \subset U, \; U \text{ open}\},$$

$$\forall E \text{ open}, \; \nu_f(E) = \sup\{\nu_f(K) : K \subset E, \; K \text{ compact}\}.$$

ν_f is called the energy measure of f associated to $(\mathcal{E}, D(\mathcal{E}))$.

Definition 2.3.3. A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ in $L^2(M,\mu)$ is called regular if there exists a subspace $\mathcal{C} \subset D(\mathcal{E}) \cap C_c(M)$ such that

1. $\mathcal{C}$ is dense in $C_c(M)$ with respect to $\|\cdot\|_\infty$.
2. $\mathcal{C}$ is dense in $D(\mathcal{E})$ with respect to $\|\cdot\|_{\mathcal{E}_1} = \big(\mathcal{E}(\cdot,\cdot) + \|\cdot\|^2_{L^2}\big)^{1/2}$.

Observation 2.3.4. For $(\mathcal{E}, D(\mathcal{E}))$ regular and any $f, g \in D(\mathcal{E})$,

$$\nu_{f,g} := \frac{1}{2}\big(\nu_{f+g} - \nu_f - \nu_g\big)$$

defines a signed measure.

We following bilinear form was introduced by Kunita in [**?**]. It measures the failure of an infinitesimal generator to satisfy the Leibniz rule.

Definition 2.3.5. Let $(\mathcal{E}, D(\mathcal{E}))$ be a D.F. in $L^2(M,\mu)$ with infinitesimal generator $L : D(L) \subset D(\mathcal{E}) \to D(\mathcal{E})$. Further, let $\mathcal{A} \subset D(\mathcal{E})$ be a subspace such that $fg \in \mathcal{A}$ for all $f, g \in \mathcal{A}$. The bilinear map

$$\Gamma(f,g) : \mathcal{A} \times \mathcal{A} \to L^2(M,\mu)$$

$$(f,g) \mapsto \frac{1}{2}\big(L(fg) - fLg - gLf\big)$$

is called the carré du champ operator associated to L.

Energy measure: $\mathcal{E}(f, fh) - \frac{1}{2}\mathcal{E}(h, f^2) = \int_M h d\nu_f, h \in C_c(M)$

Definition 2.3.6. A D.F $(\mathcal{E}, D(\mathcal{E}))$ is called local if $\forall f, g \in D(\mathcal{E})$ with compact supports such that $supp(f) \cap supp(g) = \emptyset$, then $\mathcal{E}(f,g) = 0$.
It is called strongly local if $\forall f, g \in D(\mathcal{E})$ with compact supports such that $g|_{supp(f)}$=constant , then $\mathcal{E}(f,g) = 0$.

Theorem 2.3.7 (Beurling-Deny)**.** *Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular D.F in $L^2(M,\mu)$ There exists unique positive Radon measures J,k and a unique strongly local D.F $(\mathcal{E}^c, D(\mathcal{E}^c))$ such that :*

$$\mathcal{E}(f,g) = \mathcal{E}^c(f,g) + \iint\limits_{M\times M\setminus \text{diag}} (f(x)-f(y))(g(x)-g(y))J(dx,dy) + \int_M f(x)g(x)k(dx)$$

for any $f, g \in D(\mathcal{E})$. J is called a jumping kernel measure, while k is called the killing measure.

CHAPTER 3

Dirichlet forms on fractals

3.1. Self-similar sets

Chapter 3 : DF's on fractals - (SG) Motivation : Sierpinski Gasket
Q: Does there exist "intrinsic" BM on SG?
A1: Limit of rescaled random walks
A2: Limit of DF's
3.1 Self-similar sets
(M, d) complete metric space

Definition 3.1.1. A map $\psi\colon M \mapsto M$ is called Lipschitz continuous (w.r.t. d) if

$$\text{Lip } \psi = \sup_{x,y\in M, x\neq y} \frac{d(\psi(x),\psi(y))}{d(x,y)} < \infty$$

When Lip $\psi \in (0,1)$, we call ψ a contraction map.

Remark : Since (M, d) is complete, by the Banach fixed point theorem, for any contraction $\psi : M \mapsto M$, $\exists! x \in M$ such that $\psi(x) = x$.

Proposition 3.1.2. *Let K(M)=$\{K \subseteq M\ ,\ K$ compact non-empty$\}$. For any U, V subsets of M define :*

$$\rho_M(U, V) = \inf\{\varepsilon > 0 : U \subseteq V_\varepsilon \text{ and } V \subseteq U_\varepsilon\}$$

where V_ε is defined by $V_\varepsilon = \bigcup_{x\in V} B(x, \varepsilon)$.
This metric is called the Hausdorff metric, and $(K(M), \rho_M)$ is a complete metric space.
Proof : HW

Note: The above metric on $K(M)$ is not the same as the

$$\rho^*(U,V) = \inf\{d(x,y) : x \in U, y \in V\}$$

on $K(M)$.
Indeed, pick M=$\mathbb{Z}$, $U = \{1,7\}, V = \{3,6\}$. Then $\rho^*(U,V) = 1$, but $\rho(U,V) = 2$.

Lemma 3.1.3. *For any U_1, U_2, V_1, V_2 in $K(M)$,*

$$\rho_M(U_1 \cup U_2, V_1 \cup V_2) \leq \max\{\rho_M(U_1,V_1), \rho_M(U_2,V_2)\}.$$

Proof. Let $r > max\{\rho_M(U_1,V_1), \rho_M(U_2,V_2)\}$,
Then $U_1 \subseteq (V_1)_r, V_1 \subseteq (U_1)_r$ and $U_2 \subseteq (V_2)_r, V_2 \subseteq (U_2)_r$
Therefore, $U_1 \cup U_2 \subseteq (V_1 \cup V_2)_r$ and $V_1 \cup V_2 \subseteq (U_1 \cup U_2)_r$
And thus, $\rho_M(U_1 \cup U_2, V_1 \cup V_2) \leq r$ □

Lemma 3.1.4. *Let $\psi : M \mapsto M$ be a contraction (w.r.t d). Then $\psi : K(M) \mapsto K(M)$, $U \mapsto \psi(U)$ is also a contraction (w.r.t ρ_M)*

Proof. We need to show that there exists some constant $c \in [0,1)$ such that $\rho_M(\psi(U), \psi(V)) \leq c\rho_M(U,V)$, for all $U, V \in K(M)$, so fix $U, V \in K(M)$.
Let $\varepsilon > 0$. If $U \subseteq V_\varepsilon$ and $V \subseteq U_\varepsilon$ (and thus $\rho_M(U,V) \leq \varepsilon$) ,
then $\psi(U) \subseteq \psi(V_\varepsilon) \subseteq (\psi(V))_{\varepsilon Lip(\psi)}$,
$\psi(V) \subseteq \psi(U_\varepsilon) \subseteq (\psi(U))_{\varepsilon Lip(\psi)}$
so, $\rho_M(\psi(U), \psi(V)) \leq Lip\psi \ \rho_M(U,V)$. □

Theorem 3.1.5. *Let $\{\psi_i\}_{i=1}^N$ be contractions in (M,d).*
The map $\psi : K(M) \mapsto K(M)$, $U \mapsto \cup_{i=1}^N \psi_i(U)$, has a unique fixed point, F.
(i.e. $F = \cup_{i=1}^N \psi_i(F)$).
Also, for any V in $K(M)$, $\rho_M(\psi^n(V), F) \to 0$ as n goes to infinity.

Proof. Let $U, V \in K(M)$.
By applying the preceding two lemmas, we get
$\rho_M(\psi(U), \psi(V)) = \rho_M(\cup_{i=1}^N \psi_i(U), \cup_{i=1}^N \psi_i(V)) \leq \max_{1 \leq i \leq n} \rho_M(\psi_i(U), \psi_i(V)) \leq (\max_{1 \leq i \leq n} Lip(\psi_i))\rho_M(U,V)$ □

Example 3.1.6. Consider $\psi_i : \mathbb{R}^2 \mapsto \mathbb{R}^2$, $i = 1,2,3$
$x \mapsto \frac{x-p_i}{2} + p_i$
are contractions with $Lip\psi_i = \frac{1}{2}$.
$\exists! F \in K(\mathbb{R}^2)$ such that $F = \cup_{i=1}^3 \psi_i(F)$
Approximating graphs will have vertices $V_0 = \{p_1, p_2, p_3\}$.
Let $V_n = \psi^n(V_0) = \cup_{w \in \{1,2,3\}^n} \psi_w(V_0)$

Proposition 3.1.7. *Let* $\{\psi_i\}_{i=1}^N$ *be a family of contractions in* M *with unique fixed point* $F \in K(M)$*. For any* $\{p_i\}_{i=1}^N \subset (0,1)^N$ *with* $\sum_{i=1}^N p_i = 1$*, there exists a unique probability measure on* F *such that*

$$\mu(A) = \sum_{i=1}^N p_i \mu(\psi_i^{-1}(A)) \quad \forall A \subset F \text{ Borel.} \tag{3.1.1}$$

This measure is called the self-similar measure.

Proof. Idea: apply Banach fixed point theorem on a suitable metric space of measures. Consider the space

$$\mathcal{M}_F := \{\mu : \text{Borel probability measure on } F \text{ with bounded support}\}, \tag{3.1.2}$$

endowed with the metric

$$\rho_F(\mu_1, \mu_2) := \sup\left\{ |\int_F f d\mu_1 - \int_F f d\mu_2| : \quad \text{Lip}(f) \leq 1 \text{ and } f \text{ bounded} \right\}. \tag{3.1.3}$$

HW: $(\mathcal{M}_F, \rho_F)$ *is a complete metric space.*
Define the map $\Phi : \mathcal{M}_F \to \mathcal{M}_F$ by

$$\Phi(\nu)(A) := \sum_{i=1}^N p_i \nu(\psi_i^{i-1}(A)), \quad A \subset F \text{ Borel.} \tag{3.1.4}$$

In particular, for any ν-integrable function $f : F \to \mathbb{R}$, we have

$$\int_F f d\Phi(\nu) = \sum_{i=1}^N p_i \int_F f \circ \psi_i d\nu. \tag{3.1.5}$$

By computation, Φ is a contraction:

$$\begin{aligned}
&\rho_F(\Phi(\mu_1), \Phi(\mu_2)) \\
&= \sup\left\{\left|\int_F f d\Phi(\mu_1) - \int_F f d\Phi(\mu_2)\right| : \quad \mathrm{Lip}(f) \leq 1\right\} \\
&= \sup\left\{\left|\sum_{i=1}^N p_i \left(\int_F f \circ \psi_i d\mu_1 - \int_F f \circ \psi_i d\mu_2\right)\right| : \quad \mathrm{Lip}(f) \leq 1\right\} \\
&\leq \sum_{i=1}^N p_i \sup\left\{\left|\int_F f \circ \psi_i d\mu_1 - \int_F f \circ \psi_i d\mu_2\right| : \quad \mathrm{Lip}(f) \leq 1\right\} \\
&\leq \sum_{i=1}^N p_i \mathrm{Lip}(\psi_i) \sup\left\{\left|\int_F \mathrm{Lip}(\psi_i)^{-1} f \circ \psi_i d\mu_1 - \int_F \mathrm{Lip}(\psi_i)^{-1} f \circ \psi_i d\mu_2\right| : \quad \mathrm{Lip}(f) \leq 1\right\} \\
&\leq \sum_{i=1}^N p_i \mathrm{Lip}(\psi_i) \sup\left\{\left|\int_F g d\mu_1 - \int_F g d\mu_2\right| : \quad \mathrm{Lip}(g) \leq 1\right\} \\
&\leq \max_{1\leq i\leq N} \mathrm{Lip}(\Psi_i) \cdot (\sum_{i=1}^N p_i) \cdot \rho_F(\mu_1, \mu_2) \\
&< \rho_F(\mu_1, \mu_2),
\end{aligned} \tag{3.1.6}$$

where we used the fact that $\mathrm{Lip}(\mathrm{Lip}(\psi_i)^{-1} f \circ \psi_i) \leq 1$. □

Example 3.1.8 (Sierpinski Gasket)**.** Set $\psi_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2, 3$ with $x \mapsto \frac{xp_i}{2} + p_i$, and let μ be the standard Bernoulli probability measure on SG. Then,

$$\mu(A) = \sum_{i=1}^3 \frac{1}{3}\mu(\psi_i^{-1}(A)). \tag{3.1.7}$$

Fact: μ is equivalent to the $\frac{\log(3)}{\log(2)}$-Hausdorff measure.

3.2. Convergence of Dirichlet forms

Motivation We learned that on a finite graph like (V_n, E_n) the bilinear form

$$\begin{cases} \mathcal{E}^{(n)}(f,f) := \frac{1}{2}\sum_{x\sim y}(f(x) - f(y))^2, \\ D(\mathcal{E}^{(n)}) = \ell^2(V_n, \mu_n) = \{f : V_n \to \mathbb{R} : \sum_{x\in V_n} |f(x)|, \quad \deg(x) < \infty\}, \end{cases} \tag{3.2.1}$$

defines a Dirichlet form on $\ell^2(V_n, \mu_n)$. Since $V_n \to SG$, we would wish $\mathcal{E}^{(n)} \to \mathcal{E}$.
Questions:

1. How does one understand convergence of Dirichlet forms? (When each lives in a different space)

2. Would the sequence $\{(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)})\}_{n\geq 1}$ converge?

Definition 3.2.1 (Convergence of Hilbert spaces)**.** A sequence of Hilbert spaces $\{\mathcal{H}_n\}_{n\geq 1}$ is said to converge to a Hilbert space $\mathcal{H}$ if there exists a dense subspace $\mathcal{C} \subset \mathcal{H}$ and a sequence of operators $\{\Phi_n : \mathcal{C} \to \mathcal{H}_n\}$ such that

$$\lim_{n\to\infty} \|\Phi_n f\|_{\mathcal{H}_n} = \|f\|_{\mathcal{H}}, \quad \forall f \in \mathcal{C}. \tag{3.2.2}$$

Example 3.2.2. For each $n \geq 0$ consider the measure in V_n given by

$$\mu_n = \begin{cases} 2\cdot^{-n+1} & \text{if } x \in V_n \setminus V_0 \\ 3^{-n+1} & \text{if } x \in V_0 \end{cases}. \tag{3.2.3}$$

Further, let $\mathcal{C} = C(SG) \subset L^2(SG, \mu)$ and define $\Phi_n : \mathcal{C} \to L^2(V_n, \mu_n)$, $f \mapsto f|_{V_n}$. Then,

$$\begin{aligned}\lim_{n\to\infty} \|\Phi_n f\|^2_{\ell^2(V_n,\mu_n)} &= \lim_{n\to\infty} \frac{1}{3} \sum_{x\in V_0} 3^{-n}|f(x)|^2 + \frac{2}{3} \sum_{x\in V_n\setminus V_0} 3^{-n}|f(x)|^2 \\ &=^{(\text{HW})} \lim_{n\to\infty} \frac{1}{3} \sum_{i=1}^{3} \sum_{|w|=n: w \text{ is a word}} f(\psi_w p_i)\mu(\psi_w(SG)) \\ &= \int_{SG} |f(x)|^2 d\mu.\end{aligned} \tag{3.2.4}$$

Definition 3.2.3 (Convergence)**.** Let $\mathcal{H}_n$ be a sequence of Hilbert spaces converging to a Hilbert space $\{\mathcal{H}\}_{n\geq 1}$. A sequence with $f_n \in \mathcal{H}_n$ is said to

1. strongly converge to $f \in \mathcal{H}\}$ if there exists $\{\tilde{f}_n\}_{n\geq 1} \subset \mathcal{C}$ such that

$$\|\tilde{f}_n - f\|_{\mathcal{H}} \longrightarrow 0 \quad \text{and} \quad \lim_m \limsup_n \|\Phi_n \tilde{f}_m - f_n\|_{\mathcal{H}_n} = 0. \tag{3.2.5}$$

2. weakly converge to $f \in \mathcal{H}$ if for all $\{g_n\}_{n\geq 1}$ with $g_n \in \mathcal{H}_n$ and $g_n \to g \in \mathcal{H}$ strongly, it holds that

$$\langle f_n, g_n \rangle_{\mathcal{H}_\setminus} \to \langle f, g \rangle_{\mathcal{H}}. \tag{3.2.6}$$

In particular, if $f_n \to f$ strongly, then $\|f_n\|_{\mathcal{H}_n} \to \|f\|_{\mathcal{H}}$. A sequence of bounded operators $\{L_n : D(L_n) \subset \mathcal{H}_n \to \mathcal{H}_n\}_{n\geq 1}$ is said to strongly converge to an operator $L : D(L) \subset \mathcal{H} \to \mathcal{H}$ if for any sequence $f_n \to f$ strongly, we have $L_n f_n \to Lf$ strongly.

Example 3.2.4. For any $f \in C(SG)$, the sequence $f_n := f|_{V_n}$ converge strongly to f by choosing $\tilde{f}_n \equiv f$.
For any $f \in L^2(SG, \mu)$, by density of $\mathcal{C}$, select a sequence $\{\tilde{f}_n\} \subset \mathcal{C}$ converging to f in L^2, and set $f_n = \tilde{f}_n|_{V_n}$. Then, f_n converge strongly to f.

Definition 3.2.5 (G convergence and Mosco convergence)**.** A sequence of symmetric forms $\{\mathcal{E}_n, D(\mathcal{E}_n)\}_{n\geq 1}$ is said to Γ-converge/ Mosco-converge to a symmetric form $(\mathcal{E}, D(\mathcal{E}))$ if

1. for any $\{f_n\}_{n\geq 1}(f_n \in \mathcal{H}_n)$ and $f_n \to f \in \mathcal{H}$ strongly/ weakly, we have
$$\mathcal{E}(f,f) \leq \liminf_{n\to\infty} \mathcal{E}_n(f_n, f_n), \tag{3.2.7}$$
and
2. for any $f \in \mathcal{H}$ there exists $\{f_n\}_{n\geq 1}(f_n \in \mathcal{H}_n)$ and $f_n \to f \in \mathcal{H}$ strongly and
$$\mathcal{E}(f,f) \geq \limsup_{n\to\infty} \mathcal{E}_n(f_n, f_n). \tag{3.2.8}$$

Example 3.2.6. The sequence $\{\mathcal{E}_n, \ell(V_n)\}$ where $\ell(V_n) = \{f : V_n \to \mathbb{R}\}$ Mosco converges to
$$\mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}_n(f|_{V_n}, f|_{V_n}), \quad \forall f \in C(SG). \tag{3.2.9}$$

Theorem 3.2.7. *Let $\{\mathcal{H}_n\}_{n\geq 1}$ be a sequence of Hilbert spaces that converge to a Hilbert space $\mathcal{H}$. A sequence of Dirichlet forms $\{\mathcal{E}_n, D(\mathcal{E}_n)\}_{n\geq 1}$ (on $\mathcal{H}_n$) Mosco converge to a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ (on $\mathcal{H}$) if and only if the associated semigroups $\{P_t^{(n)}\}_{t\geq 0}$ converge strongly to $\{P_t\}_{t\geq 0}$*

Proof. WLOG, set $\mathcal{H}_n = \mathcal{H}$.
($\Longrightarrow$) By Hille-Yosida theorem, it suffices to show that $R_\lambda^{(n)} f \to R_\lambda f$ strongly for any $\lambda > 0$ and $f \in \mathcal{H}$.
Fix λ and f. Since $\|R_\lambda^{(n)}\| \leq \frac{1}{\lambda}$ (by theorem 1.3.6(i)) uniformly in n, there exists a subsequence $R_\lambda^{(n_k)} \to \tilde{f}$ weakly.

1. Claim: $\tilde{f} = R_\lambda f$

   Proof. We will use the following: (HW) $R_\lambda f$ is the unique minimizer of
$$\mathcal{E}(g,g) + \lambda\langle g, g, \rangle_{\mathcal{H}} - 2\langle f, g\rangle_{\mathcal{H}}. \tag{3.2.10}$$
   This imply (for any $g \in \mathcal{H}$ and $n \geq 1$)
$$\mathcal{E}_n(g,g)+\lambda\langle g,g,\rangle_{\mathcal{H}}-2\langle f,g\rangle \geq \mathcal{E}_n(R_\lambda^{(n)} f, R_\lambda^{(n)} f)+\lambda\langle R_\lambda^{(n)} f, R_\lambda^{(n)} f\rangle-2\langle f, R_\lambda^{(n)} f\rangle. \tag{3.2.11}$$

By definition of Mosco convergence, there exists a sequence $\{g_n\}$ such that $g_n \to f$ strongly and

$$\limsup \mathcal{E}_n(g_n, g_n) \leq \mathcal{E}(f, f). \tag{3.2.12}$$

By weak convergence of $R_\lambda^{(n_k)} f$, Mosco convergence, and the above observed facts, we have

$$\begin{aligned}
&\mathcal{E}(\tilde{f}, \tilde{f}) + \lambda\langle \tilde{f}, \tilde{f}\rangle - 2\langle f, \tilde{f}\rangle \\
&\leq \liminf [\mathcal{E}_{n_k}(g, g) + \lambda\langle g, g\rangle - 2\langle f, g\rangle] \\
&\leq \limsup \mathcal{E}_n(g, g) + \lambda\langle g, g\rangle - 2\langle f, g\rangle \\
&\leq \mathcal{E}(g, g) + \lambda\langle g, g\rangle - 2\langle f, g\rangle.
\end{aligned} \tag{3.2.13}$$

Hence, taking infimum over g, we conclude that the (HW) lemma implies the desired claim. □

2. Claim: $\|R_\lambda^{(n)} f\|_{\mathcal{H}} \to \|R_\lambda f\|_{\mathcal{H}}$.

   Proof. By the (HW) lemma,

$$\mathcal{E}_n(R_\lambda^{(n)} f, R_\lambda^{(n)} f) + \lambda\langle R_\lambda^{(n)} f, R_\lambda^{(n)} f\rangle - 2\langle f, R_\lambda^{(n)} f\rangle \leq \mathcal{E}_n(g, g) + \lambda\langle g, g, \rangle_{\mathcal{H}} - 2\lambda\langle \frac{f}{\lambda}, g\rangle. \tag{3.2.14}$$

   We add the term $\lambda\langle \frac{f}{\lambda}, \frac{f}{\lambda}\rangle$ on both sides and use the Parallelogram law to infer that

$$\lambda\|R_\lambda^{(n)} f - \frac{f}{\lambda}\|^2 \leq \lambda\|g - \frac{f}{\lambda}\|^2 + \mathcal{E}_n(g, g) - \mathcal{E}_n(R_\lambda^{(n)} f, R_\lambda^{(n)} f), \tag{3.2.15}$$

   for any $g \in \mathcal{H}$. By definition of Mosco convergence, there exists a sequence $g_n \to R_\lambda f$ strongly. Replacing g with g_n above yields

$$\lambda\|R_\lambda^{(n)} f - \frac{f}{\lambda}\|^2 \leq \lambda\|g_n - \frac{f}{\lambda}\|^2 + \mathcal{E}_n(g_n, g_n) - \mathcal{E}_n(R_\lambda^{(n)} f, R_\lambda^{(n)} f), \tag{3.2.16}$$

   We take limit-supremum and use Mosco convergence definition to conclude that

$$\limsup_{n\to\infty} \lambda\|R_\lambda^{(n)} f - \frac{f}{\lambda}\|^2 \leq \lambda\|R_\lambda f - \frac{f}{\lambda}\|_{\mathcal{H}}^2. \tag{3.2.17}$$

   On the other hand, weak convergence imply the RHS is bounded by the limit-infimum of the LHS. Thus,

$$\lim_{n\to\infty} \|R_\lambda^{(n)} f - \frac{f}{\lambda}\|^2 = \|R_\lambda f - \frac{f}{\lambda}\|^2. \tag{3.2.18}$$

   We conclude the desired claim by parallelogram law in the above identity. □

Then, the above claims imply that $R_\lambda^{(n)}f$ converge strongly to $\mathcal{R}_\lambda f$ as desired.
($\Longleftarrow$) We want to prove conditions (M1) and (M2) from Definition **??**. We know that $P_t^{(n)}f \xrightarrow[n\to\infty]{s} P_tf$ for any $t\in[0,t]$, $t>0$ and $f\in\mathcal{H}$. By virtue of Hille-Yosida, the latter is equivalent to a corresponding statement for the resolvents, namely $R_\lambda^{(n)}f \xrightarrow[n\to\infty]{s} R_\lambda f$ for all $\lambda>0$ and $f\in\mathcal{H}$.

(M1) Let $f_n \xrightarrow{w} f$, i.e. $\langle f_n, g\rangle_{\mathcal{H}} \xrightarrow{n\to\infty} \langle f, g\rangle_{\mathcal{H}}$. To prove that $\liminf_{n\to\infty}\mathcal{E}_n(f,f)\geq \mathcal{E}(f,f)$, we use the characterization on $R_\lambda f$ from Exercise **??**, namely

$$\mathcal{E}(f,f)=\lim_{\lambda\to\infty}\lambda\langle (I-R_\lambda f)f,f\rangle_{\mathcal{H}}\quad \forall\, n\geq 1$$

and in particular the latter quantity its non-decreasing. Thus, for any $n\geq 1$

$$\begin{aligned}&\mathcal{E}_n(f_n,f_n)\geq \lambda\langle (I-R_\lambda^{(n)})f_n,f\rangle_{\mathcal{H}}\\ &=\lambda\langle (I-R_\lambda^{(n)})f_n,f_n\rangle_{\mathcal{H}}-\lambda\langle (I-R_\lambda^{(n)})f,f\rangle_{\mathcal{H}}+\lambda<(I-R_\lambda^{(n)})f,f\rangle_{\mathcal{H}}.\end{aligned}$$

Adding and subtracting $\lambda<(I-R_\lambda^{(n)})f,f_n\rangle_{\mathcal{H}}$ we obtain from the latter

$$\begin{aligned}\mathcal{E}_n(f_n,f_n)&\geq \lambda\langle (I-R_\lambda^{(n)})f,f\rangle_{\mathcal{H}}+\lambda\langle (I-R_\lambda^{(n)})f,f_n-f\rangle_{\mathcal{H}}\\ &+\lambda\langle (I-R_\lambda^{(n)})(f_n-f),f_n\rangle_{\mathcal{H}}\\ &\geq \lambda\langle (I-R_\lambda^{(n)})f,f\rangle_{\mathcal{H}}+2\lambda\langle (I-R_\lambda^{(n)})f,f_n-f\rangle_{\mathcal{H}}.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the inequality and using the weak convergence of $R_\lambda^{(n)}$ we finally obtain

$$\liminf_{n\to\infty}\mathcal{E}_n(f_n,f_n)\geq\xrightarrow{n\to\infty}\lambda<(I-R_\lambda)f,f\rangle_{\mathcal{H}}\xrightarrow{\lambda\to\infty}\mathcal{E}(f,f).$$

(M2) Let $f\in\mathcal{H}$. We want to find $\{f_n\}_{n\geq 1}\subset\mathcal{H}$ such that $f_n \xrightarrow[s]{n\to\infty} f$ and $\limsup_{n\to\infty}\mathcal{E}_n(f_n,f_n)\leq\mathcal{E}(f,f)$. We know

(a) $P_t^{(n)} \xrightarrow[s]{n\to\infty} P_tf$ and hence $R_\lambda^{(n)}\xrightarrow{s} R_\lambda f$ for all $\lambda>0$.

(b) A diagonal argument (HW) allows to choose a suitable subsequence for which

$$\begin{aligned}(f,f)&=\lim_{\lambda\to\infty}\langle (I-\lambda R_\lambda)f,f\rangle_{\mathcal{H}}\\ &\geq\lim_{\lambda\to\infty}\lim_{n\to\infty}\rangle(I-\lambda R_\lambda^{(n)}f,f\rangle_{\mathcal{H}}\\ &\geq\lim_{n\to\infty}\lambda_n\rangle(I-\lambda_n R_{\lambda_n}^{(n)}f,f\rangle_{\mathcal{H}},\end{aligned}$$

Define now $f_n := \lambda_n R^{(n)}_{\lambda_n} f$ for each $n \geq 1$. Then $f_n \xrightarrow[s]{n\to\infty} f$ because of (a) and

$$\begin{aligned}\mathcal{E}(f,f) &\geq \lim_{n\to\infty} \lambda_n \langle (I - \lambda_n R^{(n)}_{\lambda_n} f, f\rangle_{\mathcal{H}} \\ &\lim_{n\to\infty} \lambda_n \langle (I - \lambda_n R^{(n)}_{\lambda_n} f_n, f_n\rangle_{\mathcal{H}} + \lambda_m \|f - f_n\|^2_{\mathcal{H}} \\ &\geq \limsup_{n\to\infty} \mathcal{E}_n(f_n, f_n).\end{aligned}$$

□

Notice that in practice, checking (M1) may be difficult to check directly. Alternatively on can use the concept of *asymptotically compactness.*

Definition 3.2.8. A sequence $\{(\mathcal{E}_n, D(\mathcal{E}_n)\}_{n\geq 1}$ of Dirichlet forms (on $\mathcal{H}_n$ or $\mathcal{H}$) is said to be asymptotically compact if for any $\{f_n\}_{n\geq 1}$ with $f_n \in \mathcal{H}_n$ (or $mathcalH$) and

$$\liminf_{n\to\infty} \big(\mathcal{E}_n f_n, \mathcal{E}_n f_n\rangle_{\mathcal{H}}\big) < \infty$$

has a strongly convergent subsequence.

Theorem 3.2.9. *Let $\{(\mathcal{E}_n, D(\mathcal{E}_n))\}_{n\geq 1}$ be a sequence of Dirichlet forms on $\mathcal{H}_n$ (or $\mathcal{H}$) that is asymptotically compact. Then, the sequence Γ-converges ifa and only if it Mosco converges.*

Proof. Mosco convergence implies Γ-convergences by definition, see Definition **??**. Thus, let us assume that the sequence Γ-converges and prove the property (M2) of Mosco convergence. We argue by contradiction: Let $\{f_n\}_{n\geq 1}$ such that $f_n \xrightarrow{w} f \in \mathcal{H}$ and suppose that

$$\liminf_{n\to\infty} \mathcal{E}_n(f_n, f_n) < \mathcal{E}(f,f). \tag{3.2.19}$$

Since $f_n \xrightarrow{w} f$, we can extract a subsequence $\{f_{n_k}\}_{k\geq 1}$ with $\|f_{n_k}\|_{\mathcal{H}} \xrightarrow{n\to\infty}$. Together with (3.2.19),

$$\liminf_{k\to\infty} \mathcal{E}_{n_k}(f_{n_k}, f_{n_k}) + \|f_{n_k}\|_{\mathcal{H}} < \infty$$

and it follows from the asymptotic compactness that we may extract yet another subsequence $\{f_{n'_k}\}_{k\geq 1}$ that converges strongly. Since $f_n \xrightarrow{w} f$ by assumption, it must hold that $f_{n'_k} \xrightarrow{s} f$. However, if that is the case, by Γ-convergence it must hold that $\mathcal{E}(f,f) \leq \liminf_{k\to\infty} \mathcal{E}_{n'_k}(f_{n'_k}, f_{n'_k})$ which contradicts (3.2.19). □

3.3. Resistance forms

Our original naïve guess to define a Dirichlet form on SG was

$$\begin{cases} \mathcal{E}(f,f) := \lim\limits_{n\to\infty} \sum_{x \overset{n}{\sim} y} (f(x) - f(y))^2 \\ D(\mathcal{E}) = \overline{C(SG)}^{\mathcal{E}_1}. \end{cases} \tag{3.3.1}$$

But does this limit exist? Let us make a first computation to test whether the above seems a meaningful definition. Let us start at the approximation level $n = 0$ and consider the function $f_0 \in \ell(V_0)$ described in Figure ??. Its energy will be

$$E_0(f_0, f_0) = 1^2 + 1^2 = 2.$$

Now let us move to the approximation level $n = 1$ and consider a function $f_1 \in \ell(V_1)$ that coincides with f_0 on V_0. If we compute its standard graph energy we obtain

$$E_1(f_1, f_1) = 2(1-a)^2 + 2(a-b)^2 + 2a^2 + 2b^2.$$

One can now set up a minimization problem to find out which values of a, b minimize that expression. A little calculus gives $a = 2/5$ and $b = 1/5$. Plugging above these values,

$$E_1(f_1, f_1) = \frac{6}{5} = 2\frac{3}{5} = \frac{3}{5}E_0(f_0, f_0).$$

Repeating this procedure at level $n = 2$ we find

$$E_2(f_2, f_2) = 2\Big(\frac{3}{5}\Big)^2.$$

An educated guess now tells us

$$E_n(f_n, f_n) = \Big(\frac{3}{5}\Big)^n E_0(f_0, f_0) \xrightarrow{n\to\infty} 0$$

Well, it looks like (3.3.1) might not be a good guess...

Now what??

Definition 3.3.1. A resistance form on (M, d) is a symmetric bilinear form $(\mathcal{E}, D(\mathcal{E}))$ such that

(R1) $D(\mathcal{E})$ is a linear subspace of $\ell(M)$ that contains constants and such that $\mathcal{E}(f, f) = 0$ if and only if f is constant,

(R2) $(D(\mathcal{E})/_\sim, \mathcal{E}^{1/2})$ with $f \sim g$ if and only if $f - g \equiv \text{const}$ is a Hilbert space,

(R3) $D(\mathcal{E})$ separates points, i.e. for any $x \neq y \in M$ there exists $f \in D(\mathcal{E})$ with $f(x) \neq f(y)$,

(R4) For any $x, y \in M$,

$$R_{\mathcal{E}}(x, y) = \sup\Big\{\frac{|f(x) - f(y)|^2}{\mathcal{E}(f, f)} : f \in D(\mathcal{E}), \mathcal{E}(f, f) > 0\Big\} < \infty,$$

(R5) $(\mathcal{E}, D(\mathcal{E}))$ satisfies the Markov property, i.e. for any $f \in D(\mathcal{E})$, its unit contraction $\tilde{f}$ belongs to $D(\mathcal{E})$ and $\mathcal{E}(\tilde{f}, \tilde{f}) \leq \mathcal{E}(f, f)$.

Where does the name come from? The connection to electric network theory is apparent.

Definition 3.3.2. Let $V \subset M$ be finite. A resistance network $(\mathcal{E}, \ell(V))$ is a non-negative symmetric form given by

$$\mathcal{E}(f,g) = \langle f, -Hg \rangle_{\ell^2(V)} \qquad f,g \in \ell(V),$$

for some symmetric linear operator $H\colon \ell(V) \to \ell(V)$ with the property that $\mathcal{E}(f,f) = 0$ if and only if f is constant.

The operator H is sometimes called a *difference operator*.

Proposition 3.3.3. *Let $V \subset M$ be finite. A symmetric linear operator $H\colon \ell(V) \to \ell(V)$ defines a resistance network if and only if*

(D1) $H^\intercal = H$;

(D2) H is irreducible, i.e. for any distinct $x, y \in V$ there is a sequence $x = z_1, z_2, \ldots, z_N = y$ such that the entries $H_{z_i z_{i+1}} \neq 0$;

(D3) H has negative diagonal, i.e. $H_{xx} < 0$ for all $x \in V$;

(D4) For any distinct $x, y \in V$, $H_{xy} \geq 0$.

Proof. Homework. □

With the notation above, we my think of an electric network where $V \subset M$ corresponds to the set of nodes or terminals, H^{-1}_{xy} is the resistance between the nodes x and y, and for any potential $f \in \ell(V)$, the quantity Hf corresponds to the current associated with f.

In the context of electric networks there is a natural concept of distance, so-called *effective resistance*, not to confuse with the resistance between two nodes.

Definition 3.3.4. Let $V \subset M$ be finite and $(\mathcal{E}, \ell(V))$ be a resistance network. The associated effective resistance distance is defined as

$$R_{\mathcal{E}}(x,y) := \big(\min\{\mathcal{E}(f,f)\colon f \in \ell(V), f(x) = 0, f(y) = 1\}\big)^{-1}$$

for any $x, y \in V$.

Example 3.3.5. The standard graph energy on any approximation level of the Sierpinski gasket V_n defines a resistance network. Indeed one can check that

$$\mathcal{E}(f,g) = \sum_{x \overset{n}{\sim} y} (f(x) - f(y))(g(x) - g(y))$$

$$
\begin{aligned}
&= \sum_{x\in V_n} g(x)\Big(\sum_{y\overset{n}{\sim}x}(f(x)-f(y)\Big) \\
&= -\langle g, H_n f\rangle_{\ell^2(V_n)}
\end{aligned}
$$

For the case $n=1$ we can write explicitly the operator H_1

$$
H_1 = \begin{pmatrix}
-2 & 0 & 0 & 1 & 1 & 0 \\
0 & -2 & 0 & 1 & 0 & 1 \\
0 & 0 & -2 & 0 & 1 & 1 \\
1 & 1 & 0 & -4 & 1 & 1 \\
1 & 0 & 1 & 1 & -4 & 1 \\
0 & 1 & 1 & 1 & 1 & -4
\end{pmatrix}
$$

In order to construct a non-trivial limit form it will be useful to understand when two networks are "electrically equivalent".

Definition 3.3.6. Let $V_1 \subset V_2 \subset M$ be finite. Two resistance networks $(\mathcal{E}_1, \ell(V_1))$ and $(\mathcal{E}_2, \ell(V_2))$ are called compatible if for all $f \in \ell(V_1)$

$$\mathcal{E}_1(f,f) = \min\{\mathcal{E}_2(g,g)\colon g \in \ell(V_2), g|_{V_1} \equiv f\}.$$

The function that attains the minimum above is called the *harmonic extension* of f to V_2.

Example 3.3.7. Consider the standard graph energy of the two first approximations of the Sierpinski gasket, $V_0 \subset V_1$ from Figure **??**,

$$
\begin{aligned}
E_0(f,f) &= \sum_{x\overset{0}{\sim}y}(f(x)-f(y))^2, \qquad f \in \ell(V_0) \\
E_1 f,f) &= \sum_{x\overset{1}{\sim}y}(f(x)-f(y))^2, \qquad f \in \ell(V_1).
\end{aligned}
$$

Given $f \in \ell(V_0)$ as in Figure **??**, we ask ourselves how to extend f to a function $g \in \ell(V_1)$ that minimizes $E_1(g,g)$. Calling x, y, z the unknown values of g in $V_1 \setminus V_0$, and solving the corresponding (quadratic) minimization problem one arrives at

$$
\begin{aligned}
x &= \frac{2}{5}a + \frac{2}{5}b + \frac{1}{5}c \\
y &= \frac{2}{5}b + \frac{2}{5}c + \frac{1}{5}a \\
z &= \frac{2}{5}a + \frac{2}{5}c + \frac{1}{5}b.
\end{aligned}
$$

This is usually called the "$\frac{2}{5}$-$\frac{1}{5}$ rule". Plugging into $E_1(g,g)$ we obtain

$$E_1(g,g) = \frac{3}{5}E_0(f,f).$$

Thus, defining

$$\mathcal{E}_0(f,f) := E_1(f,f), \qquad f \in \ell(V_0)$$
$$\mathcal{E}_1(f,f) := \frac{5}{3}E_1(f,f), \qquad f \in \ell(V_1)$$

makes these networks compatible.

Proposition 3.3.8. *Let $V \subset M$ finite and $(\mathcal{E}, \ell(V))$ a resistance network with operator $H : \ell(V) \to \ell(V)$. For any $U \subset V$ let*

$$H = \left(\begin{array}{c|c} T_U & J_U^T \\ \hline J_U & X_U \end{array}\right).$$

Then:

(i) $\widetilde{\mathcal{E}}(f,g) := -\langle f, X_U g\rangle_{\ell^2(V\setminus U)}$ *defines a resistance network in* $V \setminus U$*.*

(ii) The resistance network $(\mathcal{E}_U, \ell(U))$ *given by*

$$\mathcal{E}_U(f,g) = -\langle f, (T_U - J_U^T X_U^{-1} J_U)g\rangle_{\ell^2(U)}$$

is compatible with $(\mathcal{E}, \ell(V))$*.*

Proof. (i) Extend $f, g \in \ell(V \setminus U)$ by zero on U.

(ii) (a) Using characterization of $H_U := T_U - J_U^T X_U^{-1} J_U$ from Prop 3.3.3 we get that $(\mathcal{E}_U, \ell(U))$ defines a resistance network.

(b) For compatibility with $(\mathcal{E}, \ell(V))$ we need to check that

$$\mathcal{E}_U(f,f) = \min\{\mathcal{E}(g,g) : g \in \ell(V),\ g|_U = f\}, \quad \forall f \in \ell(U).$$

Observe that for $g \in \ell(V)$ with $f = g|_U$ we have

$$\begin{aligned}
\mathcal{E}(g,g) &= -\langle g, Hg\rangle_{\ell(V)} = -\begin{pmatrix} g|_U & g|_{V\setminus U}\end{pmatrix}\begin{pmatrix} T_U & J_U^T \\ J_U & X_U\end{pmatrix}\begin{pmatrix} g|_U \\ g|_{V\setminus U}\end{pmatrix} \\
&= \langle g|_U, -T_U g|_U\rangle - \langle g|_U, J_U^T g|_{V\setminus U}\rangle - \langle g|_{V\setminus U}, J_U g|_U\rangle \\
&\quad - \langle g|_{V\setminus U}, X_U g|_{V\setminus U}\rangle \qquad (\pm\langle g|_U, (J_U^T X_U^{-1} J_U) g|_U\rangle) \\
&= -\langle g|_U, T_U g|_U\rangle + \langle g|_U, (J_U^T X_U^{-1} J_U) g|_U\rangle \\
&\quad - \langle J_U g|_U, g|_{V\setminus U}\rangle - \langle X_U^{-1} J_U g|_U, J_U g|_U\rangle \\
&\quad - \langle g|_{V\setminus U}, J_U g|_U\rangle - \langle g|_{V\setminus U}, X_U g|_{V\setminus U}\rangle \\
&= -\langle g|_U, (T_U - J_U^T X_U^{-1} J_U) g|_U\rangle \\
&\quad - \langle X_U^{-1} J_U g|_U, X_U g|_{V\setminus U}\rangle - \langle X_U^{-1} J_U g|_U, J_U g|_U\rangle \\
&\quad - \langle g|_{V\setminus U}, X_U g|_{V\setminus U} + J_U g|_U\rangle
\end{aligned}$$

$$
\begin{aligned}
&= -\langle g|_U, (T_U - J_U^T X_U^{-1} J_U) g|_U\rangle \\
&\quad - \langle X_U^{-1} J_U g|_U, X_U g|_{V\setminus U} + J_U g|_U\rangle - \langle g|_{V\setminus U}, X_U g|_{V\setminus U} + J_U g|_U\rangle \\
&= -\langle g|_U, (T_U - J_U^T X_U^{-1} J_U) g|_U\rangle \\
&\quad - \langle g|_{V\setminus U} + X_U^{-1} J_U g|_U, X_U g|_{V\setminus U} + J_U g|_U\rangle \\
&= -\langle g|_U, (T_U - J_U^T X_U^{-1} J_U) g|_U\rangle \\
&\quad - \langle g|_{V\setminus U} + X_U^{-1} J_U g|_U, X_U g|_{V\setminus U} + J_U g|_U\rangle \\
&= -\langle g|_U, (T_U - J_U^T X_U^{-1} J_U) g|_U\rangle \\
&\quad - \langle g|_{V\setminus U} + X_U^{-1} J_U g|_U, X_U (g|_{V\setminus U} + X_U^{-1} J_U g|_U)\rangle \\
&= \mathcal{E}_U(g|_U, g|_U) + \widetilde{\mathcal{E}}(g|_{V\setminus U} + X_U^{-1} J_U g|_U,\ g|_{V\setminus U} + X_U^{-1} J_U g|_U).
\end{aligned}
$$

Since $\widetilde{\mathcal{E}}(g|_{V\setminus U} + X_U^{-1} J_U g|_U,\ g|_{V\setminus U} + X_U^{-1} J_U g|_U) \geq 0$, for $f = g|_U$, we have

$$\mathcal{E}_U(f, f) = \min\{\mathcal{E}(g, g) : g \in \ell(V),\ g|_U = f\}$$

and the minimum is attained for

$$g = \begin{cases} f, \text{ in } U \\ X_U^{-1} J_U f, \text{ in } V \setminus U. \end{cases}$$

□

Corollary 3.3.9. *Let $(\mathcal{E}, \ell(V))$ and $(\widetilde{\mathcal{E}}, \ell(U))$ with $U \subset V$. These resistance networks are compatible if and only if*

$$\widetilde{\mathcal{E}}(f, f) = -\langle f, (T_U - J_U^T X_U^{-1} J_U) f\rangle_{\ell^2(U)},$$

for

$$H = \begin{pmatrix} T_U & J_U^T \\ J_U & X_U \end{pmatrix}$$

being the associated operator of $(\mathcal{E}, \ell(V))$.

Example 3.3.10. On the triangle, V_0, with vertices p_1, p_2, p_3 and resistances between them 1, we have

$$\mathcal{E}_0(f, g) = -\langle f, H_0 g\rangle,$$

with

$$H_0 = \left(\begin{array}{cc|c} -2 & 1 & 1 \\ 1 & -2 & 1 \\ \hline 1 & 1 & -2 \end{array}\right).$$

The effective resistance between p_1, p_2 is

$$R_{\mathcal{E}_0}(p_1, p_2) = (\min\{\mathcal{E}_0(f, f) : f(p_1) = 0, f(p_2) = 1\})^{-1}$$

$$
\begin{aligned}
&= (\mathcal{E}_0(\delta_{p_2}, \delta_{p_2}))^{-1} \\
&= (\langle \delta_{p_2}, (T_U - J_U^T X_U J_U)\delta_{p_2}\rangle)^{-1} \\
&= \left[\begin{pmatrix} 0 & 1 \end{pmatrix} \left[\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} 2^{-1} \begin{pmatrix} 1 & 1 \end{pmatrix}\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]^{-1} \\
&= \left[\begin{pmatrix} 0 & 1 \end{pmatrix} \frac{3}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]^{-1} = \frac{2}{3},
\end{aligned}
$$

where $U = \{p_1, p_2\}$ and $\delta_{p_2} = \begin{cases} 0, & \text{on } p_1 \\ 1, & \text{on } p_2 \end{cases}$.

Lemma 3.3.11. *Let $(\mathcal{E}, \ell(V))$ and $(\widetilde{\mathcal{E}}, \ell(U))$ with $U \subset V$ be compatible resistance networks. Then,*

$$R_{\mathcal{E}}(x, y) = R_{\widetilde{\mathcal{E}}}(x, y), \quad \forall x, y \in U.$$

Proof.

$$
\begin{aligned}
R_{\mathcal{E}}(x, y) &= (\min\{\mathcal{E}(f, f) : f \in \ell(V), f(x) = 0, f(y) = 1\})^{-1} \\
&= (\min\{\min\{\mathcal{E}(f, f) : f \in \ell(V), f|_U = g\} : g(x) = 0, g(y) = 1\})^{-1} \\
&= (\min\{\widetilde{\mathcal{E}}(g, g) : g \in \ell(U), g(x) = 0, g(y) = 1\})^{-1} \\
&= R_{\widetilde{\mathcal{E}}}(x, y).
\end{aligned}
$$

□

Theorem 3.3.12. *Let $(\mathcal{E}, \ell(V))$ be a resistance network. The effective resistance:*

$$R_{\mathcal{E}}(x, y) := (\min\{\mathcal{E}(f, f) : f \in \ell(V), f(x) = 0, f(y) = 1\})^{-1}$$

defines a metric in V.

Proof. For the triangle inequality: Let $U = \{x, y, z\} \subset V$, $\widetilde{H} = T_U - J_U^T X_U^{-1} J_U$, $\widetilde{H}_{xy}^{-1}$ the resistance between x, y, and $H = (H_{xy})_{x,y \in V} = \begin{pmatrix} T_U & J_U^T \\ J_U & X_U \end{pmatrix}$. Then, by Lemma 3.3.11, we have

$$R_{\mathcal{E}}(x, y) = R_{\widetilde{\mathcal{E}}}(x, y) = (\langle \delta_y, \widetilde{\widetilde{H}} \delta_y \rangle)^{-1} = \cdots = \frac{\widetilde{H}_{xy}^{-1}(\widetilde{H}_{xz}^{-1} + \widetilde{H}_{yz}^{-1})}{\widetilde{H}_{xy}^{-1} + \widetilde{H}_{xz}^{-1} + \widetilde{H}_{yz}^{-1}}.$$

□

Thus far we have only considered a finite number (one or two) of resistance forms. However ultimately we want to consider a sequence o approximating forms and its limit in the following sense.

Definition 3.3.13. Let $V_0 \subset V_1 \subset \ldots \subset V_m \subset \ldots \subset M$ be a sequence of finite sets. The sequence of resistance networks $\{(\mathcal{E}_m, \ell(V_m)\}_{m\geq 0}$ is said to be compatible if $(\mathcal{E}_m, \ell(V_m))$ and $(\mathcal{E}_{m+1}, \ell(V_{m+1}))$ are compatible for all $m \geq 0$.

Definition 3.3.14. The limit of a compatible sequence of resistance forms $\{(\mathcal{E}_m, \ell(V_m)\}_{m\geq 0}$ is the form $(\mathcal{E}_*, D(\mathcal{E}_*))$ defined as

$$\begin{cases} \mathcal{E}_*(f,f) := \lim_{m\to\infty} \mathcal{E}_m(f|_{V_m}, f|_{V_m}), \\ D(\mathcal{E}_*) := \{f \in \ell(V_*) \colon \mathcal{E}_*(f,f) < \infty\}, \end{cases}$$

where $V_* := \bigcup_{m\geq 0} V_m$.

Remark 3.3.15. By definition of compatibility, for any $f \in D(\mathcal{E}_*$

$$\mathcal{E}_m(f|_{V_m}, f|_{V_m}) = \min\{\mathcal{E}_{m+1}(g,g) \colon g|_{V_m} = f, g \in \ell(V_{m+1}\} \leq \mathcal{E}_{m+1}(f|_{V_{m+1}}, f|_{V_{m+1}}).$$

Thus, the sequence $\{\mathcal{E}_m(f|_{V_m}, f|_{V_m})\}_{m\geq 0}$ is monotone increasing and hence the limit $\mathcal{E}_*(f,f)$ is well-defined.

Lemma 3.3.16. *Let $(\mathcal{E}_*, D(\mathcal{E}_*))$ be the limit of compatible sequence of resistance networks. The associated effective resistance distance satisfies*

$$R_{\mathcal{E}_*}(x,y) = \max\Big\{\frac{|f(x)-f(y)|^2}{\mathcal{E}_*(f,f)} \colon f \in D(\mathcal{E}_*), \quad f(x) \neq f(y)\Big\}.$$

Proof. Let $f \in D(\mathcal{E}_*)$ and note that for any $g = af + b$ with $a, b \in \mathcal{R}$,

$$\frac{|g(x)-g(y)|^2}{\mathcal{E}_*(g,g)} = \frac{a^2|f(x)-f(y)|^2}{a^2\mathcal{E}_*(f,f)} = \frac{|f(x)-f(y)|^2}{\mathcal{E}_*(f,f)}.$$

If $f(x) \neq f(y)$, writing $g(z) = \frac{f(z)-f(x)}{f(y)-f(x)}$ we have $g(x) = 0$, $g(y) = 1$ and

$$\begin{aligned} &\max\Big\{\frac{|f(x)-f(y)|^2}{\mathcal{E}_*(f,f)} \colon f \in D(\mathcal{E}_*), \quad f(x) \neq f(y)\Big\} \\ &= \max\Big\{\frac{1}{\mathcal{E}_*(g,g)} \colon g \in D(\mathcal{E}_*), \quad g(x) = 0, g(y) = 1\Big\} \\ &= \big(\min\{\mathcal{E}_*(g,g) \colon g \in D(\mathcal{E}_*), \quad g(x) = 0, g(y) = 1\}\big)^{-1} = R_{\mathcal{E}_*(x,y)} \end{aligned}$$

□

Theorem 3.3.17. *Let $(\mathcal{E}_*, D(\mathcal{E}_*))$ be the limit of compatible sequence of resistance networks.*

(i) *$\mathcal{E}_*$ is a non-negative bilinear form, any $f \in D(\mathcal{E}_*)$ is continuous with respect to the effective resistance metric and $\mathcal{E}_*(f,f) = 0$ if and only if f is constant;*

(ii) *For any* $f,g\in D(\mathcal{E}_*)$ *let* $f\sim g$ *if and only if* $f-g\equiv \text{const}$. *Then* $(D(\mathcal{E}_*)/_\sim,\mathcal{E}_*^{1/2})$ *is a Hilbert space;*

(iii) *For any finite set* $V\subset V_*$ *and* $f\in\ell(V)$ *there exists* $f_*\in D(\mathcal{E}_*)$ *such that* $f_*|_V\equiv f$.

(iv) *For any* $x,y\in V_*$

$$\max\Big\{\frac{|f(x)-f(y)|^2}{\mathcal{E}_*(f,f)}\colon f\in D(\mathcal{E}_*),\quad f(x)\neq f(y)\Big\}$$

defines a metric in V_**;*

(v) $(\mathcal{E}_*,D(\mathcal{E}_*))$ *satisfies the Markov property.*

Proof. Let $\{(\mathcal{E}_m,\ell(V_m)\}_{m\geq 0}$ denote the sequence of compatible networks.

(i) We prove continuity (in fact Hölder continuity. By virtue of Lemma 3.3.16, for any $x,y\in V_*$ and $f\in D(\mathcal{E}_*)$

$$|f(x)-f(y)|^2=\mathcal{E}_*(f,f)\frac{|f(x)-f(y)|^2}{\mathcal{E}_*(f,f)}\leq \mathcal{E}_*(f,f)R_{\mathcal{E}_*}(x,y).$$

(ii) We prove completeness of the space. First note that by fixing $x_0\in V_*$, the space $\mathcal{F}_0:=f\in D(\mathcal{E}_*)\colon f(x_0)=0\}$ equipped with the norm $\mathcal{E}_*^{1/2}$ is isomorphic to $(D(\mathcal{E}_*)/_\sim,\mathcal{E}_*^{1/2})$. We thus show that $(\mathcal{F}_0,\mathcal{E}_*^{1/2})$ is complete. Let $\{f_n\}_{n\geq 1}\subset\mathcal{F}_0$ be Cauchy.
For any $g\in\ell(V_m)$, let $h_m(g)\in\ell(V_*)$ denote the harmonic extension of g to V_*, that is the function such that

$$\mathcal{E}_*(h_m(g),h_m(g))=\min\{\mathcal{E}_*(f,f)\colon f\in D(\mathcal{E}_*),\ f|_{V_m}\equiv g\}.$$

Due to the compatibility of the sequence of networks, $\mathcal{E}_m(g,g)=\mathcal{E}_*(h_m(g),h_m(g)$ for all $g\in\ell(V_m)$. In this manner, $\mathcal{E}_*$ defines an inner product on $\ell_0(V_m):=\{g\in\ell(V_m)\colon g(x_0)=0\}$ for any m large enough so that $x_0\in V_m$. Thus, since $\{f_n\}_{n\geq1}$ is $\mathcal{E}_*$-Cauchy, for any such large m we have

$$\mathcal{E}_*(f_k|_{V_m},f_l|_{V_m})\leq\mathcal{E}_*(f_k,f_l)\xrightarrow{k,l\to\infty}0$$

whence $\{f_n|_{V_m}\}_{n\geq1}$ is $\mathcal{E}_m$-Cauchy and therefore convergent in $\ell_0(V_m)$. Let $g^m\in\ell_0(V_m)$ denote the limit of this function, i.e.

$$\mathcal{E}_m(g^m-f_k|_{V_m},g^m-f_k|_{V_m}\xrightarrow{k\to\infty}0\qquad(3.3.2)$$

and in particular $g^{m+1}|_{V_m} = g^m$ because $\mathcal{E}_m(g^m - g^{m+1}|_{V_m}, g^m - g^{m+1}|_{V_m}) = 0$. Therefore, there is $g \in \ell_0(V_m)$ such that $g|_{V_m} = g^m$ for all m large enough.
We now need to check that $g \in D(\mathcal{E})$. By definition,

$$\begin{aligned}\mathcal{E}_m(g|_{V_m}, g|_{V_m}) &= \mathcal{E}_m(g^m, g^m)\\ &= \mathcal{E}_m(g^m - f_k|_{V_m}, g^m - f_k|_{V_m}) + 2\mathcal{E}_m(g^m - f_k|_{V_m}, f_k|_{V_m}) \qquad (3.3.3)\\ &+ \mathcal{E}_m(f_k|_{V_m}, f_k|_{V_m})\\ &< \sup_{k,m} \mathcal{E}_m(f_k|_{V_m}, f_k|_{V_m}),\end{aligned}$$

where the terms in (3.3.3) can be made arbitrarily small using Cauchy-Schwarz and (3.3.2).
Finally, we claim that $\mathcal{E}_*(g - f_n, g - f_n) \xrightarrow{n\to\infty} 0$. Indeed, let $\varepsilon > 0$. First, we can choose $n > 0$ such that

$$\mathcal{E}_*(f_n - f_k, f_n - f_k) < \varepsilon \qquad \forall\, k > n. \tag{3.3.4}$$

Now we may choose M such that

$$|\mathcal{E}_*(g - f_n, g - f_n) - \mathcal{E}_m(g^m - f_n|_{V_m}, g^m - f|_{V_m})| < \varepsilon \forall\, m > M \tag{3.3.5}$$

and $k > 0$ so that

$$|\mathcal{E}_m(g^m - f_n|_{V_m}, g^m - f|_{V_m}) - \mathcal{E}_m(f_n|_{V_m} - f_k|_{V_m}, f|_{V_m}) - f_k|_{V_m}| < \varepsilon \qquad \forall\, k > n. \tag{3.3.6}$$

(iii) Let $V \subset V_*$ be finite and $f \in \ell(V)$. Then there exists $M > 0$ such that $V \subset V_m$ for all $m \geq M$. First we can extend f to V_m by defining $f_M \equiv f$ on V and $f \equiv 0$ on $V_m \setminus V$. We can then take the harmonic extension $\bar{f} = h_M(f_M)$, which necessarily satisfies

$$\mathcal{E}_*(\bar{f}, \bar{f}) \leq \mathcal{E}(f_M, f_M) \leq \infty.$$

(iv) Combine Theorem 3.3.12 and Lemma 3.3.16.

(v) Let $f \in D(\mathcal{E}_*)$. Then its unit contraction $\tilde{f}$ satisfies

$$\begin{aligned}\mathcal{E}_*(\tilde{f}, \tilde{f}) &:= \lim_{m\to\infty} \mathcal{E}_m(\tilde{f}|_{V_m}, \tilde{f}|_{V_m}) = \lim_{m\to\infty} -\langle H_m \tilde{f}|_{V_m}, \tilde{f}|_{V_m}\rangle_{\ell^2(V_m)}\\ &= \lim_{m\to\infty} \sum_{y \overset{n}{\sim} x} (H_m)^{-1}_{xy} (\tilde{f}(x) - \tilde{f}(y))^2 \leq \lim_{m\to\infty} \sum_{y \overset{n}{\sim} x} (H_m)^{-1}_{xy} (f(x) - f(y))^2.\end{aligned}$$

□

Corollary 3.3.18. *Let $(\mathcal{E}_*, D(\mathcal{E}_*))$ be the limit of a compatible sequence of resistance networks $\{(\mathcal{E}_m, \ell(V_m))\}_{m\geq 0}$ with $V_* = \bigcup_{m\geq 0} V_m$. Then $(\mathcal{E}_*, D(\mathcal{E}_*))$ is a resistance form on $(\overline{V_*}^{R_{\mathcal{E}_*}}, R_{\mathcal{E}_*})$.*

Proof. Any $f \in D(\mathcal{E}_*)$ is Hölder continuous with respect to $R_{\mathcal{E}_*}$. We may then extend f to $\overline{V_*}^{R_{\mathcal{E}_x}}$ by density. □

Example 3.3.19. We would like to construct $(\mathcal{E}_*, D(\mathcal{E}_*))$ on $(SG, R_{\mathcal{E}_*})$. If we write $SG = \bigcup_{i=1}^{3} \psi_i(SG)$ and denote the three corner points of SG by p_1, p_2, and p_3, we may define

$$V_0 = \{p_1, p_2, p_3\}, \quad \text{and} \quad V_m = \bigcup_{i=1}^{3} \psi_i(V_{m-1}), \quad m \geq 1. \tag{3.3.7}$$

Then we have that for $V_* = \bigcup_{i=1}^{3} \psi_i(V_{m-1})$, $\overline{V_*}^{d_E} = SG$. If we can show that the topologies induced by d_E and $R_{\mathcal{E}_*}$ coincide, then V_* will be dense with respect to $R_{\mathcal{E}_*}$ and we may apply Corollary 3.3.18.
To show this, first let $x, y \in V_m$ be neighbors at level m, so that $d_E(x,y) = \left(\frac{1}{2}\right)^m$. Let h_y be the harmonic extension of

$$\delta_y(z) = \begin{cases} 1, & z = y \\ 0, & x \in V_* \setminus \{y\}. \end{cases}$$

Note that $\delta_y(z) \in \ell(V_m)$. By harmonicity, we have

$$\begin{aligned} \mathcal{E}_*(h_y, h_y) &= \lim_{n\to\infty} \mathcal{E}_n(h_y|_{V_n}, h_y|_{V_n}) = \mathcal{E}_m(\delta_y(z), \delta_y(z)) \\ &= \sum_{y \overset{m}{\sim} x} \left(\frac{5}{3}\right)^m (\delta_y(x) - \delta_y(y))^2 \\ &= \begin{cases} 4\left(\frac{5}{3}\right)^m & \text{if } y \in V_m \setminus V_0 \\ 2\left(\frac{5}{3}\right)^m & \text{if } y \in V_0. \end{cases} \end{aligned}$$

On the one hand, by definition, $R_{\mathcal{E}_*}(x,y)$ is a maximum over all functions $f \in D(\mathcal{E}_*)$ with $f(x) = 0$ and $f(y) = 1$ of, so

$$\frac{|h_y(x) - h_y(y)|}{\mathcal{E}_*(h_y, h_y)} \leq R_{\mathcal{E}_*}(x,y).$$

On the other hand, h_y minimizes $\mathcal{E}_*(f,f)$ for any $f \in D(\mathcal{E}_*)$ with $f(x) = 0$ and $f(y) = 1$ of, so

$$\frac{|h_y(x) - h_y(y)|}{\mathcal{E}_*(h_y, h_y)} = \frac{1}{\mathcal{E}_*(h_y, h_y)} \geq R_{\mathcal{E}_*}(x,y).$$

We see that

$$R_{\mathcal{E}_*}(x,y) \approx \left(\frac{5}{3}\right)^m = d_E(x,y)^{\frac{\log(5/3)}{\log(2)}}.$$

Note that the exponent here is $d_w - d_H$, where $d_w = \frac{\log(5)}{\log(2)}$ is the walk dimension and $d_H = \frac{\log(3)}{\log(2)}$ is the Hausdorff dimension. This relationship holds for neighboring points at level m. For any two points, we may construct a chain of such neighboring points which connects the two, so in fact, this relation holds for all points. Hence, the topologies agree, so $\overline{V_*}^{d_E} = \overline{V_*}^{R_{\mathcal{E}_*}} = SG$ and we may apply Corollary 3.3.18.

Theorem 3.3.20. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on $(M, R_{\mathcal{E}_*})$ and assume $(M, R_{\mathcal{E}_*})$ is separable. Further, let μ be a σ-finite Borel measure on $(M, R_{\mathcal{E}_*})$ and*

$$\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + \int_M fg d\mu.$$

If $\mathcal{F} \cap L^2(M, d\mu)$ is dense in $L^2(M, \mu)$ with respect to $\| \cdot \|_{L^2(M,\mu)}$, then there exists a non-positive self-adjoint operator $L : D(L) \subset L^2(M, \mu) \to L^2(M, \mu)$ such that

$$\mathcal{E}(f, g) = -\langle L^{1/2} f, L^{1/2} g\rangle_{L^2(M,\mu)}.$$

In particular, $(\mathcal{E}, \mathcal{F} \cap L^2(M, d\mu))$ is a Dirichlet Form on $(M, R_{\mathcal{E}_*})$.

We use the following characterization.

Theorem 3.3.21. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a densely defined symmetric non-negative bilinear form in $L^2(M, \mu)$. The following are equivalent:*

(i) There exists a non-positive self-adjoint $L : D(L) \to L^2(M, \mu)$ such that $\mathcal{E}(f, g) = -\langle L^{1/2} f, L^{1/2} g\rangle_{L^2(M,\mu)}$

(ii) $(D(\mathcal{E}), \mathcal{E}_1^{1/2})$ is a complete (Hilbert) space.

Proof. See *Heat Kernels and Spectral Theory* by E. B. Davies. □

Proof of Theorem 3.3.20. First, we show that the space $(D(\mathcal{E}), \mathcal{E}_1^{1/2})$ is complete, where $\mathcal{E}_1^{1/2}(f, f) = \varepsilon(f, f) + \int_M f^2 d\mu$.
Let $\{f_n\}_{n\geq 1} \subseteq D(\mathcal{E})$ be $\mathcal{E}_1$-Cauchy. Then,

$$\mathcal{E}(f_n - f_m, f_n - f_m) \to 0 \text{ as } n, m \to \infty, \quad (*)$$
$$\|f_n - f_m\|_{L^2} \to 0 \text{ as } n, m \to \infty. \quad (**)$$

① Take $x_0 \in M$ and set $g_n = f_n - f_n(x_0)$, then by property (R2) of resistance forms, $\exists g \in \mathcal{F}$ with $g(x_0) = 0$ and such that $\varepsilon(g_n - g, g_n - g) \to 0$ as $n, m \to \infty$.

② By virtue of (R4) in the definition of resistance forms,

$$|g_n(x) - g(x)|^2 = |(g_n - g)(x) - (g_n - g)(x_0))|^2 \leq \mathcal{E}(g_n - g, g_n - g) R_{\mathcal{E}}(x, x_0).$$

③ Since μ is σ-finite there exists a sequence of bounded sets $\{M_m\}_{m\geq 0}$ with $0 < \mu(M_m) < \infty$, and $\bigcup_m M_m = M$. Therefore, by ②, $\|g_n|_{M_m} - g|_{M_m}\| \xrightarrow{n\to\infty} 0$, $\forall m \geq 0$, which means $g_n|_{M_m}$ is $L^2(M_m, \mu|_{M_m})$-Cauchy.

We also need that $\{f_n(x_0)\}_{n\geq 1}$ converges to something, but it follows from (**) that $(-g_n|_{M_m} + f_n|_{M_m})$ $(= f_n(x_0))$ is Cauchy in $L^2(M_m, \mu|_{M_m})$ hence $\exists c \in \mathbb{R}$ such that $f_n(x_0) \xrightarrow{n\to\infty} c$.

④ Define $f = g + c$ and check :

ⓐ $\mathcal{E}(f - f_n, f - f_n) \xrightarrow{n\to\infty} 0$

ⓑ $\|f - f_n\|_{L^2} \xrightarrow{n\to\infty} 0.$

To prove ⓐ,

$$\begin{aligned}\mathcal{E}(f - f_n, f - f_n) &= \mathcal{E}(g + c - g_n - f_n(x_0), g + c - g_n - f_n(x_0)) \\ &= \mathcal{E}(g - g_n, g - g_n) + 2\varepsilon(g - g_n, c - f_n(x_0)) + \mathcal{E}(c - f_n(x_0), c - f_n(x_0)).\end{aligned}$$

ⓑ We know $f_n|_{M_m} = g_n|_{M_m} + f_n(x_0) \xrightarrow[n\to\infty]{L^2(M_m,\mu)} g|_{M_m} + c = f|_{M_m}$. Thus, by (**) there exists $\widetilde{f} \in L^2(M, \mu)$ such that $||f_n - \widetilde{f}||_{L^2(M,\mu)} \xrightarrow[n\to\infty]{} 0$, and thus

$\widetilde{f}|_{M_m} = f|_{M_m}$ for all m.

□

3.4. Singularity of energy measure on SG

Let us recall that $(\mathcal{E}, D(\mathcal{E})$ is a local and regular Dirichlet form on SG. For any $f \in D(\mathcal{E})$, the energy measure of f was defined as the measure ν_f such that

$$\int_M h d\nu_f = \mathcal{E}(f, fh) - \frac{1}{2}\mathcal{E}(f^2, h) = -\langle f, L(fh)\rangle + \frac{1}{2}\langle L(f^2), h\rangle$$

$\forall h \in C_c(M)$, and notice that $\int_M d\nu_f = \mathcal{E}(f, f)$. When existent, the associated Carré du champ operator satisfies

$$\Gamma(f, f) = \frac{1}{2}L(f^2) - fL(fh)$$

so that

$$\int_M h\Gamma(f, f)d\mu = \langle h, \Gamma(f, f)\rangle = \frac{1}{2}\langle L(f^2), h\rangle - \langle fL(fh), h\rangle.$$

The main theorem of this is section is:

Theorem 3.4.1. *For any $f \in D(\mathcal{E}) \subseteq L^2(\mathrm{SG}, \mu)$ the energy measure ν_f is singular w.r.t μ.*

Before proving it, we recall some basics about the construction of SG. First,

$$\mathrm{SG} = \bigcup_{i=1}^{3} \psi_i(\mathrm{SG}) = \Psi(\mathrm{SG}).$$

Second, we define W_m to be the space of words $\{1,2,3\}^m$, and $W_* := \bigcup_{i=1}^{\infty} W_m$. In this way, the graphs that approximate SG have vertex set

$$V_m = \bigcup_{w=W_m} \psi_{w_1} \circ \psi_{w_2} \circ \ldots \circ \psi_{w_m}(V_0).$$

Finally, SG is equipped with the standard Bernulli measure which satisfies

$$\mu(SG) = \sum_{i=1}^{3} \mu(\psi_i(SG)).$$

Note that $\mu(\psi_i(SG)) = \frac{1}{3}$.

Proposition 3.4.2. *For any $x \in \mathrm{SG}$, there exists $w \in W_1^{\mathbb{N}}$, not necessarily unique, such that*

$$x = \bigcap_{m\geq 0} \psi_{w_1\ldots w_m}(SG) = \lim_{m\to\infty} \psi_{w_1\ldots w_m}(SG).$$

Proof. This follows from the fact that we have a monotone decreasing sequence of compact sets, $\psi_{w_1\ldots w_m}(SG) = \psi_{w_1\ldots w_{m-1}}\psi_{w_m}(SG) \subseteq \psi_{w_1\ldots w_{m-1}}(SG) \subset SG$. □

Corollary 3.4.3. *For any $x \in SG$, there exists $\{T_n\}_{n\geq 0}$, $T_n = \psi_{w_1\ldots w_m}(SG)$, such that $d(x, T_n) \xrightarrow{n\to\infty} 0$.*

Lemma 3.4.4. *$(SG, \{T_n^x\}_{n\geq 1, x\in SG}, \mu)$ is a probability space.*

The proof of this is left as an exercise, following from the fact that $\{T_n^x \colon n \geq 1\}_{x\in SG}$ is a basis for the topology of SG.

The next Lemma will use the Lebesgue differentiation theorem for doubling measures. In the case of SG we have something stronger: there is $c > 0$ such that $c^{-1}r^\alpha \leq \mu(B(x,r)) \leq cr^\alpha$ for some exponent α. Indeed, if $r > 0$ and n satisfies $2^{-n} < r \leq 2^{-n+1}$, then

$$\frac{1}{3} r^{\frac{\log 3}{\log 2}} \leq 2^{-n\frac{\log 3}{\log 2}} = \frac{1}{3^n} = \mu(T_n^x) \leq \mu(B(x,r)) \leq \mu(T_{n-1}^x) \leq 3r^{\frac{\log 3}{\log 2}}.$$

A measure satisfying this property is called $\frac{\log 3}{\log 2}$-Ahlfors regular because there is a constant $C > 0$ such that

$$C^{-1} d(x,y)^{\frac{\log 3}{\log 2}} \leq \mu(B(x,r)) \leq C d(x,y)^{\frac{\log 3}{\log 2}}$$

for any $x, y \in \mathrm{SG}$.

Lemma 3.4.5. *Let $f \in D(\mathcal{E})$ with associated energy measure ν_f. Assume that for μ-a.e. $x \in SG$ and $\{T_n^x\}$,*

$$\lim_{n\to\infty} 3^n \nu_f(T_n^x) = 0.$$

Then $\nu_f \perp \mu$.

Proof. Suppose $\nu \ll \mu$. Then, there is a measurable function h such that for μ-a.e. $x \in SG$,

$$0 = \lim_{n\to\infty} \frac{\nu_f(T_n^x)}{\mu(T_n^x)} = \lim_{n\to\infty} \frac{1}{\mu(T_n^x)} \int_{T_n^x} h d\mu = h(x),$$

where the first equality follows from assumption and the last by Lebesgue differentiation. □

The strategy to prove theorem 3.4.1 (DOUBLE CHECK NUMBERING) is to first show lemma 3.3.23 for harmonic functions (i.e. it minimizes $\mathcal{E}(f,f)$), and then extend to any function.

Lemma 3.4.6. *There exist matrices $A_1, A_2, A_3 \in \mathcal{M}_{3\times 3}$ such that for any harmonic function h,*

$$h|_{\psi_i(V_0)} = A_i h|_{V_0}. \qquad (*)$$

In particular, $\dim\{h \in D(\mathcal{E}) : h$ *is harmonic*$\} = 3$ *and* $A_1^2 + A_2^2 + A_3^2 = \frac{3}{5}I$.

Proof. A basis for the space of harmonic functions is given by $\{h_1, h_2, h_3\}$ where each of these functions is defined to be 1 in a different vertex of V_0 and 0 elsewhere. Therefore the space of harmonic functions is 3-dimensional. By applying the "$\frac{2}{5}, \frac{1}{5}$ rule", one finds the matrices

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 1/5 & 2/5 \end{bmatrix},\ A_2 = \begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 0 & 1 & 0 \\ 1/5 & 2/5 & 2/5 \end{bmatrix},\ A_3 = \begin{bmatrix} 2/5 & 1/5 & 2/5 \\ 1/5 & 2/5 & 2/5 \\ 0 & 0 & 1 \end{bmatrix}$$

satisfying $(*)$ and then the equality $A_1^2 + A_2^2 + A_3^2 = \frac{3}{5}I$ follows by a direct computation. □

Corollary 3.4.7. *There exist matrices $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \in \mathcal{M}_{2\times 2}$ such that any non constant harmonic function satisfies*

$$h|_{\psi_i(V_0)} = \tilde{A}_i h|_{V_0}.$$

Proof. One can construct a basis for this subspace by choosing $\{\tilde{h}_1, \tilde{h}_2\}$ as linear combinations of the basis specified in the previous lemma so that constants are not included in their span. Then, $\tilde{A}_i$ is obtained by expressing A_i with respect to this new basis. □

We proceed to the proof of theorem 3.4.1 (NUMBER MAY CHANGE – DOUBLE-CHECK).

Proof. The goal is to show that for $x \in SG$ and $\{T_n^x\}$ an approximating sequence of n-cells, $3^n \nu_h(T_n) \to 0$ as $n \to \infty$ whenever h is harmonic and non constant. It suffices to show that

$$\frac{1}{n}\log(3^n \nu_n(T_n^x)) < 0$$

for large n. The reason of this is we can use a result in random matrix theory due to Frustenberg and Kesten, which states that for $Y_1, Y_2, \dots$ an i.i.d. sequence of $n \times n$ matrices with $\mathbb{E}[(\log Y_i)_+] < \infty$, there exists $\gamma \neq 0$ such that $\lim_{n\to\infty} \frac{1}{n} \log \|Y_1 \cdots Y_n\| = \gamma$ almost surely.

Step 1: To x there corresponds a word $w_1 w_2 \dots$ which gives an approximate sequence of triangular cells $T_n^x = \psi_{w_1 w_2 \dots w_n}(SG)$. By virtue of Corollary 3.4.7, any harmonic function h may be approximated by

$$h|_{T_n} = A_{w_n} \cdots A_{w_1} h|_{V_0}.$$

Step 2: Note that

$$\begin{aligned}
3^n \nu_h(T_n^x) &= 3^n \sum_{\substack{z \overset{n}{\sim} y \\ z,y \in T_n^x}} \left(\frac{5}{3}\right)^n (h(z) - h(y))^2 \\
&= 3^n \left(\frac{5}{3}\right)^n \mathcal{E}_0(h \circ \psi_{w_1 \dots w_n}, h \circ \psi_{w_1 \dots w_n}) \\
&= 3^n \left(\frac{5}{3}\right)^n \langle -H_0 A_{w_n} \cdots A_{w_1} h, A_{w_n} \cdots A_{w_1} h \rangle_{\ell^2} \\
&= 3^n \left(\frac{5}{3}\right)^n \|(-H_0)^{1/2} A_{w_n} \cdots A_{w_1} h\|_{\ell^2}^2 \\
&= 5^n \|(-H_0)^{1/2} A_{w_n} \cdots A_{w_1} h\|_{\ell^2}^2.
\end{aligned}$$

Step 3: Now,

$$\begin{aligned}
\|(-H_0)^{1/2} h|_{V_0}\|_{\ell^2}^2 &= \mathcal{E}_0(h, h) \\
&= \sum_{i=1}^{3} \left(\frac{5}{3}\right) \mathcal{E}_0(h \circ \psi_i, h \circ \psi_i) \\
&= 5 \sum_{i=1}^{3} \int_{\psi_i(SG)} \left\|(-H_0)^{1/2} A_i(x) h|_{V_0}\right\|_{\ell^2}^2 d\mu(x) \\
&= 5 \int_{SG} \left\|(-H_0)^{1/2} A_{w_1}(x) h|_{V_0}\right\|_{\ell^2}^2 d\mu(x) \\
&= \dots = 5^n \int_{SG} \left\|(-H_0)^{1/2} A_{w_n} \cdots A_{w_1}(x) h|_{V_0}\right\|_{\ell^2}^2 d\mu(x).
\end{aligned}$$

Step 4: Lastly observe that

$$\begin{aligned}
&\int_{SG} \log \|(-H_0)^{1/2} A_{w_n} \cdots A_{w_1}(x) h|_{V_0}\|_{\ell^2} d\mu(x) \\
&< \log \int_{SG} \|(-H_0)^{1/2} A_{w_n} \cdots A_{w_1}(x) h|_{V_0}\|_{\ell^2} d\mu(x)
\end{aligned}$$

$$\leq \frac{1}{2}\log\left[\int_{SG}\|(-H_0)^{1/2}A_{w_n}\cdots A_{w_1}(x)h|_{V_0}\|_{\ell^2}^2 d\mu(x)\right]$$
$$= \frac{1}{2}\log 5^{-n}\|(-H_0)^{1/2}h|_{V_0}\|_{\ell^2}^2$$

where the first and second inequalities follow by Jensen and Cauchy-Schwartz respectively, and the last equality follows by step 3. So by Frustenberg-Kesten,

$$\lim_{n\to\infty}\frac{1}{n}\log\|(-H_0)^{1/2}A_{w_n}\cdots A_{w_1}h|_{V_0}\| = \gamma < \frac{1}{2}\log 5^{-1}.$$

□